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STUDIES ON
DIVERGENT SERIES AND SUMMABILITY

BY

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To My Father

SYLVESTER FORD

This Book is Gratefully Dedicated.

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PREFACE

During the academic year 1908–9 the author was privileged to give as a part of his work at the University of Michigan a course of lectures on infinite series, with especial reference in the second semester to *divergent* series — a subject which, despite the uncertain value so long attached to it, seemed clearly to be coming into increasing prominence and importance in mathematical analysis. Little was accomplished, however, as regards divergent series beyond the merest beginning; yet this was sufficient to awaken a desire to continue farther and this in turn resulted in a course being given throughout the whole of the following year devoted entirely to divergent series and the related topic of summability. But this year also closed with much less ground satisfactorily covered than had been expected, unforeseen difficulties having arisen from time to time, some due to the inherent complexities of the subject in hand and others to the somewhat hastily conceived and hence unsatisfactory state in which much of the related literature was found to be. Thus the course still seemed altogether incomplete. It was therefore decided to continue it once more throughout the following year, 1910–11, and indeed for a like reason it was finally continued throughout 1911–12. As the lectures and class-room discussions progressed, permanent notes were kept in the hope that the whole might possibly pass through the press at some future time and appear in book form — a hope which, after various delays during which the original notes have been considerably supplemented, now reaches its realization in the appearance of the present volume. In its final form it certainly presents a large mass of detail and is doubtless open to criticism in many respects, but it does not seem advisable to attempt any further defence for it than is contained in the remaining sections of this preface wherein, after certain generalities, the content and motive of the various chapters are discussed in some detail.

Speaking roughly, the study of divergent series, at least as the author has come to conceive of it, may be divided into two parts, the one concerning the so-called *asymptotic series* and the other the *theory of summability*. Of these the first, representing the older aspect, originated in an isolated note by CAUCHY in 1843[1] relating to the well-known series of Stirling for $\log \Gamma(x)$, viz.:

$$(1)\quad \log \Gamma(x) = \tfrac{1}{2}\log 2\pi + (x - \tfrac{1}{2})\log x - x + \frac{B_1}{1\cdot 2}\frac{1}{x} - \frac{B_2}{3\cdot 4}\frac{1}{x^3} + \frac{B_3}{5\cdot 6}\frac{1}{x^6} - \cdots.$$

$(B_m = m\text{th Bernoulli number.})$

CAUCHY pointed out that this series, though divergent for all values of x, may be

[1] "Sur l'emploi légitime des séries divergentes," *Compt. Rend. de l'Acad. des Sciences*, Vol. 17, pp. 370–376.

used in computing $\log \Gamma(x)$ when x is large (and positive)—in fact, it was shown that, having fixed the number n of terms taken, the absolute error committed by stopping the summation at the nth term is less than the absolute value of the next succeeding term, and hence becomes arbitrarily small ($n > 3$) with increasing x. CAUCHY's work on divergent series was confined, however, to the single series (1) and, owing to the emphasis placed upon convergent processes exclusively by the successors of CAUCHY and ABEL, no further progress was made in this interesting field until the subject at last reappeared after more than forty years in connection with the researches of POINCARÉ upon the irregular solutions of linear differential equations.[2] POINCARÉ considered those divergent series (normal series) of the form

$$(2) \qquad e^{f(x)}x^{\rho}(a_0 + a_1/x + a_2/x^2 + \cdots); \qquad \begin{array}{l} f(x) = \textit{polynomial in } x, \\ \rho = \textit{constant}. \end{array}$$

which for some time had been known to satisfy *formally* linear differential equations of certain types having the point $x = \infty$ as an "irregular" point, and he showed essentially that in general to every such formal solution there corresponds an *actual* solution which can be represented by (2) in much the same sense as (1) was described above as representing $\log \Gamma(x)$.[3] In view of the important significance of such results both from the standpoint of the possible use of divergent series as well as from that of the theory of differential equations, POINCARÉ set apart and discussed in some detail a broad class of divergent series of the special form (2), applying to them the name of "asymptotic series." POINCARÉ's results, however, in so far as they concerned differential equations, were noticeably incomplete, being limited by certain unfortunate restrictions, and thus his original studies have given rise in later years to numerous researches, notably by HORN, in which noteworthy advances have been made, though open questions in this connection still remain. Corresponding investigations (likewise begun by POINCARÉ) pertaining to linear difference equations have been undertaken in recent years and carried to an advanced stage by HORN, NÖRLUND, and others. Meanwhile an important aspect of the theory of asymptotic series has come into view, especially in England under the leadership of BARNES and HARDY; namely, that of actually determining the asymptotic developments of a given function—a problem of decided interest for the study and classification of functions in general. This latter aspect of the subject presents a high degree of complexity and doubtless has made hardly more than a beginning at the present time. In fact, it has thus far been approached only by confining the attention to a very limited number of special functional types.[4]

[2] "Sur les intégrales irrégulières des équations linéaires," *Acta Math.*, Vol. 8 (1886), pp. 259–344. Mention should be made also of STIELTJES who simultaneously with POINCARÉ resumed the study of divergent series, confining his attention, however, to the computational aspects of certain special series. (Thesis, *Ann. de l'Éc. Nor.* (3), Vol. 3 (1886), p. 201.)

[3] For the more accurate statements, see Chap. III.

[4] For details, see Chap. II.

The theory of summability, or second general aspect of divergent series mentioned above, is essentially concerned with the question as to whether in any proper sense a " sum " may be assigned to the series, assumed divergent,

$$\sum_{n=0}^{\infty} a_n. \tag{3}$$

This question has been scientifically attacked only within comparatively recent years, the most common avenue of approach being through the so-called boundary-value (Grenzwert) problem in the theory of analytic functions.[5] Thus FROBENIUS, without having in view the study of divergent series, showed in the first place that if one has a power series whose radius of convergence is equal to 1:

$$\sum_{n=0}^{\infty} a_n x^n; \qquad r = \textit{radius of convergence} = 1 \tag{4}$$

and writes $s_n = a_0 + a_1 + \cdots + a_n$, then

$$\lim_{x=1-0} \sum_{n=0}^{\infty} a_n x^n = \lim_{n=\infty} \frac{s_0 + s_1 + \cdots + s_n}{n+1} \tag{5}$$

whenever the indicated limit on the right exists.[6] Now, the first member of (5) is naturally associated with the corresponding series (3) (in general divergent) obtained by placing $x = 1$ in (4). Thus, at least if one confines the attention to divergent series (3) of the particular type just mentioned, it becomes natural to assign sums in accordance with the formula

$$s = \lim_{n=\infty} \frac{s_0 + s_1 + \cdots + s_n}{n+1}, \tag{6}$$

whenever the indicated limit exists. Moreover, this formula finds additional justification in the demonstrable fact that for any *convergent* series (3) the sum, regarded in the ordinary sense, viz., $s = \lim_{n=\infty} s_n$, agrees with that given by (6)—*i. e.*, formula (6) is *consistent*. Aside from this one formula (6) many others are now known which serve with more or less appropriateness to define the sum of a divergent series, both when the series is of the special type above mentioned and when otherwise. To what ultimate extent these formulas *are* appropriate, how far the theories of summability erected upon them serve any justifiable purpose in analysis, whether the different sums thus assigned involve mutual inconsistencies—these and other questions may well be asked and more will be said on this point presently.[7] Suffice it to say here that formula (6) has been found in

[5] For an elementary description of the problem, see JAHRAUS, "Das Verhalten der Potenzreihen auf dem Konvergenzkreise historisch-kritisch dargestellt." Program des Gymnasiums Ludwigshafen (1901), pp. 1–56. See also KNOPP, "Grenzwerte von Reihen bei der Annäherung an die Konvergenzgrenze." Dissertation, Berlin, 1907.

[6] "Ueber die Leibnitzsche Reihe," *Jour. für Math.*, Vol. 89 (1880), pp. 262–264.

[7] Interesting comments by PRINGSHEIM relative to such questions are to be found in Vol. I of the "Encyklopädie der math. Wissenschaften," §§ 39–40.

particular to yield interesting and valuable results when applied to Fourier series and the other important allied developments in mathematical physics — developments in terms of Bessel functions, Legendre functions, etc. Such applications alone go far toward assuring a permanent place in analysis to the theory of summability as now commonly understood.

Turning now more specifically to the contents of the present volume, Chapter I considers certain aspects of the so-called Maclaurin Sum-Formula, the especial aim being to develop and summarize into actual theorems those results which are of importance in this connection to the study of divergent series. These when once obtained are of particular service in the problem of determining the asymptotic developments of a given function, and it is to this that Chapter II is then devoted. Beginning with very easy illustrative studies, the Chapter proceeds to problems of greater and greater difficulty and eventually treats the general problem already considered by various investigators of determining the asymptotic developments of the general integral (entire) function of rank p (order > 0), following which, at the close of the chapter, the problem of determining the asymptotic developments of functions defined by power series is briefly considered. Chapter III concerns the asymptotic solutions of linear differential equations and is an attempt to summarize briefly and without proof what are deemed to be the most essential results thus far known in this field, with mention also of the corresponding results obtainable in the study of linear difference equations, and with indications as to certain open questions still remaining in both connections. Chapter IV considers the theory of summability with the especial attempt, as in previous chapters, to single out what seems most essential. More specifically, it makes an examination of a few of the standard definitions of "sum" with the idea of subjecting each to a number of tests which, as the author has come to view the subject, every such definition should satisfy. For example, it is well known that if a really logical general theory of summability is ever to be constructed it cannot include all definitions of sum that satisfy merely the condition of *consistency* (§ 37) since this alone does not insure uniqueness of sum. Therefore, observing the genesis of the whole subject from the boundary value problem as described above, it is proposed to arbitrarily limit the general theory to those series (3) for which the corresponding power series (4) has a radius of convergence equal to 1 and then retain only such definitions of sum as give the *unique* value

$$s = \lim_{x=1-0} \sum_{n=0}^{\infty} a_n x^n.$$

Definitions which do this are said to satisfy the *boundary value condition* (§ 39). Such definitions not only all give the same sum to a given series (convergent or divergent) (3), but they at once serve a useful purpose in analysis from the fact that they frequently come to furnish the analytic continuation of the series (4)

over some portion of its circle of convergence, or indeed in some cases, as in the definitions of BOREL, throughout regions lying entirely outside that circle. However limited the scope of a general theory of summability as thus conceived, it at least has perfect definiteness and logical coherence and finds immediate usefulness in the theory of functions of a complex variable, and we venture the opinion that some such characteristics as these must be preserved in any general theory of summability that is to retain a permanent place in analysis.[8] No attempt will be made here to describe the other tests which Chapter IV sets up, but it should be remarked that only a few of the standard definitions of sum are tested out since they suffice to illustrate the spirit of the undertaking. The chapter closes with a brief account of absolutely summable series and a statement of certain supplementary theorems and corollaries upon summability in general.

A most important aspect of the theory of summability, as the author regards it, lies in its applications mentioned above to Fourier series and other allied developments in mathematical physics, and this forms the subject of Chapter V. For the sake of completeness the treatment is made to include both convergence and summability. It is based upon a general method for the study of all such developments due to DINI and appearing, though in somewhat diffuse and inaccessible form, in his great work entitled "Serie di Fourier e altre rappresentazioni analitiche delle funzioni di una variabile reale" (Pisa, 1880). DINI naturally considered at the time of his investigations only the question of convergence (not including uniform convergence), but his methods are here shown to be readily extended so as to be applicable to studies in summability. Especial effort has been made here as in the other chapters to summarize all essential conclusions from time to time into actual theorems.

To Professor Alexander Ziwet the author would here express his deep gratitude. Not only has the book enjoyed the benefits of his critical judgment in many ways, but his sympathy and kindly interest have served as a constant encouragement, and indeed they are responsible in no small measure for the appearance of the whole in its present form. The author is much indebted also to his colleagues Professors C. E. Love and Tomlinson Fort, the former for various suggestions and criticisms, and the latter for the valuable aid he has rendered in reading the proofs.

ANN ARBOR,
April, 1915

[8] The adoption of any one definition for "summable series" evidently involves the excluding of many series previously classed as summable; yet we believe the time has arrived when a single universal definition should if possible be agreed upon, however disastrous its immediate effects upon one or more of the special forms of definition now current. The present situation in this matter is strikingly analogous to the state of confusion which led CAUCHY and ABEL to the formulation of their universal definition for "convergent series," notwithstanding the exclusions brought about and the consequent objections urged by contemporary mathematicians.

CHAPTER I

THE MACLAURIN SUM-FORMULA, WITH INTRODUCTION TO THE STUDY OF ASYMPTOTIC SERIES

1. The following formula (Maclaurin Sum-Formula[1])

$$\sum_{x=a}^{b-h} f(x) = \frac{1}{h}\int_a^b f(x)dx - \tfrac{1}{2}[f(b) - f(a)] + \frac{B_1 h}{2!}[f'(b) - f'(a)]$$

$$(1) \qquad - \frac{B_2 h^3}{4!}[f'''(b) - f'''(a)] + \cdots + \frac{(-1)^m B_{m-1} h^{2m-3}}{(2m-2)!}[f^{(2m-3)}(b) - f^{(2m-3)}(a)]$$

$$+ \cdots; \qquad B_m = mth\ Bernoulli\ number$$

plays an important part in the modern theory of divergent series and we shall therefore begin by pointing out certain facts (cf. Theorems I, II, III and IV) connected with its legitimate use. These will form the basis of the studies undertaken in Chapter II.

Following the discussion of (1), we shall also give in the present Chapter (cf. §§ 13–17) an outline of the general theory of asymptotic series as originally developed by Poincaré in his classical memoir in the *Acta Mathematica* (1886), the elements of this theory being likewise needed for the proper development of Chapter II.

2. In order to carry out the desired studies relative to the formula (1), let us begin by supposing that there is given any function u_x (real or complex) of the real variable x which, together with its first $2m+1$ derivatives, is continuous within a certain interval (a, b). For any value of x such that $a \leqq x \leqq x + h < b$ ($h = constant$) we may then write

$$(2) \qquad \Delta u_x \equiv u_{x+h} - u_x = hu_x' + \frac{h^2}{2!}u_x'' + \cdots + \frac{h^{2m}}{(2m)!}u_x^{(2m)} + \int_0^h \frac{(h-z)^{2m}}{(2m)!}u_{x+z}^{(2m+1)}dz,$$

as appears directly upon applying an integration by parts $2m$ times to the last term in the second member.

More generally, it appears in like manner that when $0 \leqq k \leqq 2m - 1$ we may write

[1] Known also as the Euler sum-formula. For comments upon the historical aspect of the subject, see Barnes, *Proceedings London Math. Soc.* (2), Vol. 3 (1905), p. 253.

2

$$\Delta u_x^{(k)} = hu_x^{(k+1)} + \frac{h^2}{2!}u_x^{(k+2)} + \cdots + \frac{h^{2m-k}}{(2m-k)!}u_x^{(2m)} + \int_0^h \frac{(h-z)^{2m-k}}{(2m-k)!}u_{x+z}^{(2m+1)}dz, \tag{3}$$

while the corresponding formula for the case $k = 2m$ is

$$\Delta u_x^{(2m)} = \int_0^h u_{x+z}^{(2m+1)}dz. \tag{4}$$

Whence if $H_0, H_1, H_2, \cdots H_{2m}$ be the $2m + 1$ constants determined by the equations

$$\begin{cases} H_0 = 1, \qquad H_{2m} = 0, \\ H_k + \dfrac{H_{k-1}}{2!} + \dfrac{H_{k-2}}{3!} + \cdots + \dfrac{H_1}{k!} + \dfrac{H_0}{(k+1)!} = 0; \qquad 1 \leqq k \leqq 2m-1 \end{cases} \tag{5}$$

we shall have

$$-hu_x' + \sum_{k=0}^{2m} H_k h^k \Delta u_x^{(k)} = \int_0^h u_{x+z}^{(2m+1)} \sum_{k=0}^{2m} \frac{H_k h^k (h-z)^{2m-k}}{(2m-k)!}dz$$

or

$$hu_x' = \sum_{k=0}^{2m} H_k h^k \Delta u_x^{(k)} + r_m(x, h) \tag{6}$$

where

$$r_m(x, h) = -\int_0^h u_{x+z}^{(2m+1)} \sum_{k=0}^{2m} \frac{H_k h^k (h-z)^{2m-k}}{(2m-k)!}dz. \tag{7}$$

Formula (6) bears a close relation, as we shall see, to the Maclaurin Sum-Formula (1).

We first proceed to determine the values of the constants H_k, noting certain changes which thereby become possible in the form of (7).

If we place

$$\varphi_{2m}(z) = \frac{H_0 z^{2m}}{(2m)!} + \frac{H_1 h z^{2m-1}}{(2m-1)!} + \frac{H_2 h^2 z^{2m-2}}{(2m-2)!} + \frac{H_4 h^4 z^{2m-4}}{(2m-4)!} + \cdots + \frac{H_{2m-2}h^{2m-2}z^2}{2!} + H_{2m}h^{2m} \tag{8}$$

and

$$\psi_{2m}(z) = \frac{H_3 h^3 z^{2m-3}}{(2m-3)!} + \frac{H_5 h^5 z^{2m-5}}{(2m-5)!} + \cdots + \frac{H_{2m-1}h^{2m-1}z}{1!} \tag{9}$$

we have

$$r_m(x, h) = -\int_0^h u_{x+z}^{(2m+1)}[\varphi_{2m}(h-z) + \psi_{2m}(h-z)]dz.$$

Let us now develop $\varphi_{2m}(h-z) + \psi_{2m}(h-z)$ in ascending powers of z. We obtain

$$\varphi_{2m}(h-z)+\psi_{2m}(h-z)=\sum_{k=0}^{2m}H_k\frac{h^k(h-z)^{2m-k}}{(2m-k)!}$$

$$=\sum_{k=0}^{2m}H_k\sum_{j=0}^{2m-k}(-1)^j\frac{h^{2m-j}z^j}{j!(2m-k-j)!}=\sum_{j=0}^{2m}(-1)^j\frac{h^{2m-j}z^j}{j!}\sum_{k=0}^{2m-j}\frac{H_k}{(2m-k-j)!}.$$

But from (5) we have

$$\sum_{k=0}^{2m-j}\frac{H_k}{(2m-k-j)!}=0+H_{2m-j}=H_{2m-j}\quad(j\neq 2m-1);\qquad \sum_{k=0}^{1}\frac{H_k}{(1-k)!}=-H_1.$$

Hence,

$$\varphi_{2m}(h-z)+\psi_{2m}(h-z)=\frac{H_0z^{2m}}{(2m)!}+\frac{H_1hz^{2m-1}}{(2m-1)!}+\sum_{j=0}^{2m-2}(-1)^jH_{2m-j}\frac{h^{2m-j}z^j}{j!}$$

$$=\varphi_{2m}(z)-\psi_{2m}(z).$$

Thus, if we place $z=h/2$ we obtain

$$\psi_{2m}\left(\frac{h}{2}\right)=-\psi_{2m}\left(\frac{h}{2}\right);\qquad \psi_{2m}\left(\frac{h}{2}\right)=0.$$

The last relation, however, cannot exist for all positive integral values of m unless the coefficients of the various terms of $\psi_{2m}(z)$ are each equal to zero. Hence,

$$H_3=H_5=H_7=\cdots=H_{2m-1}=0, \tag{10}$$

and we obtain the relations

$$r_m(x,h)=-\int_0^h u_{x+z}^{(2m+1)}\varphi_{2m}(h-z)dz, \tag{11}$$

$$\varphi_{2m}(h-z)=\varphi_{2m}(z). \tag{12}$$

As to the coefficients $H_1, H_2, H_4, H_6, \cdots H_{2m-2}$, we have[2]

$$H_1=-\tfrac{1}{2},\qquad H_{2r}=\frac{(-1)^{r+1}}{(2r)!}B_r;\qquad r=1,2,3,\cdots(m-1) \tag{13}$$

where B_r is the rth Bernoulli number.

3. We now proceed to establish the two following properties of the functions $\varphi_{2m}(z)$ (commonly known in case $h=1$ as the functions of Bernoulli):

(*a*) "The function $\varphi_{2m}(z)$ does not change sign between $z=0$ and $z=h$ and is positive in this domain when m is even and negative when m is odd."

(*b*) "The expression $|\varphi_{2m}(z)|$ when considered for values of z between $z=0$ and $z=h$ has its maximum value at $z=h/2$."

[2] This result like others which concern the well-known properties of the Bernoulli numbers and functions, will here be presupposed. For a proof, see Malmsten, *Journ. für Math.*, Vol. 35 (1847), p. 64.

For the proof of (a) let us consider the expression

$$\varphi'_{2m-2}(z) = \frac{z^{2m-3}}{(2m-3)!} + \frac{H_1 h z^{2m-4}}{(2m-4)!} + \frac{H_2 h^2 z^{2m-5}}{(2m-5)!} + \frac{H_4 h^4 z^{2m-7}}{(2m-7)!} + \cdots + \frac{H_{2m-6} h^{2m-6} z^3}{3!} + H_{2m-4} h^{2m-4} z.$$

Supposing for the moment that this is positive whenever $0 < z < h/2$, let us multiply it by $h^{-2m}dh$ and integrate from $h = h$ to $h = +\infty$. The result, except for the factor h^{-2m+1}, is

$$\frac{z^{2m-3}}{(2m-3)!\,(2m-1)} + \frac{H_1 z^{2m-4} h}{(2m-4)!\,(2m-2)} + \frac{H_2 z^{2m-5} h^2}{(2m-5)!\,(2m-3)} + \cdots + \frac{H_{2m-6} z^3 h^{2m-6}}{3!\,5} + \frac{H_{2m-4} z h^{2m-4}}{1!\,3},$$

and this must likewise be positive when $0 < z < h/2$. Let us now multiply the last expression by dz and integrate from $z = z$ to $z = h/2$. We obtain

$$\left[\frac{\varphi'_{2m}(z)}{z} - H_{2m-2}h^{2m-2}\right]_z^{h/2} = -\frac{\varphi'_{2m}(z)}{z} + \frac{\varphi'_{2m}\left(\frac{h}{2}\right)}{\frac{h}{2}}.$$

But $\varphi'_{2m}(h/2) = 0$, as follows from (12). We therefore conclude that if $\varphi'_{2m-2}(z)$ is positive when $0 < z < h/2$, then $\varphi'_{2m}(z)$ is negative throughout the same domain. Now, $\varphi_2(z) = z^2/2 - zh/2$, $\varphi'_2(z) = z - h/2$. Whence, $\varphi'_{2m}(z)$, considered for values of z within the indicated interval, will be positive or negative according as m is even or odd, while the expression

$$\varphi_{2m}(z) = \int_0^z \varphi_{2m}'(z)dz$$

will be positive if m is even and negative if m is odd. It follows from (12) that $\varphi_{2m}(z)$ has the indicated properties for the interval $0 < z < h$.

Concerning (b) we note that if m is even we have shown that $\varphi_{2m}'(z)$ is positive when $0 < z < h/2$ and that $\varphi'_{2m}(h/2) = 0$. Moreover, since

$$\varphi'_{2m}(z) = -\varphi'_{2m}(h - z),$$

the same function is negative when $h/2 < z < h$. Thus, the statement in question follows from elementary considerations in the theory of maxima and minima. Likewise we reach the same result when m is odd, since $\varphi'_{2m}(z)$ is then negative from $z = 0$ to $z = h/2$ and positive from $z = h/2$ to $z = h$.

4. These results being established, we return to formula (6). In this formula let us take

$$u_x = \int_a^x f(x)dx,$$

where $f(x)$ together with its first $2m$ derivatives is continuous from $x = a$ to $x = b$. Then u_x together with its first $2m + 1$ derivatives will be continuous within the same interval so that for any value of x for which $a \leqq x < x + h \leqq b$ we shall have (cf. (5), (11), (13))

$$
\begin{aligned}
hf(x) = \int_x^{x+h} f(x)dx - \frac{h}{2}[f(x+h) - f(x)] + \frac{B_1h^2}{2!}[f'(x+h) - f'(x)] \\
- \frac{B_2h^4}{4!}[f'''(x+h) - f'''(x)] + \cdots \\
+ \frac{(-1)^m B_{m-1}h^{2m-2}}{(2m-2)!}[f^{(2m-3)}(x+h) - f^{(2m-3)}(x)] + r_m(x, h),
\end{aligned}
\tag{14}
$$

where

$$r_m(x, h) = -\int_0^h f^{(2m)}(x+z)\varphi_{2m}(z)dz. \tag{15}$$

Let us now suppose that $b - a$ is an integral multiple of h, *i. e.*, $b - a = nh$ and allow x to take successively the values a, $a + h$, $a + 2h$, $\cdots$ $a + (n-1)h$. By adding the corresponding results (14) and dividing by h we obtain

$$
\begin{aligned}
\sum_{q=0}^{n-1} f(a + qh) = \sum_{x=a}^{b-h} f(x) = \frac{1}{h}\int_a^b f(x)dx - \tfrac{1}{2}[f(b) - f(a)] \\
+ \frac{B_1h}{2!}[f'(b) - f'(a)] - \frac{B_2h^3}{4!}[f'''(b) - f'''(a)] + \cdots + \cdots \\
+ \frac{(-1)^m B_{m-1}h^{2m-3}}{(2m-2)!}[f^{(2m-3)}(b) - f^{(2m-3)}(a)] + R_m,
\end{aligned}
\tag{16}
$$

where

$$R_m = -\frac{1}{h}\int_0^h \sum_{x=a}^{b-h} f^{(2m)}(x+z)\varphi_{2m}(z)dz. \tag{17}$$

By placing $m = \infty$ *we thus arrive at formula* (1) *provided, however, that*

$$\lim_{m=\infty} R_m = 0.$$ [3]

5. We proceed to consider certain properties of the remainder R_m corresponding to the cases in which $f(x)$ is real. From result (a) of § 3 we may apply the

[3] For noteworthy cases in which this condition is fulfilled, see Markoff's "Differenzenrechnung" (Leipzig, 1896), Chap. 9, § 8.

first law of the mean for integrals and write

$$R_m = \frac{-1}{h}\sum_{x=a}^{b-h} f^{(2m)}(x+\theta h)\int_0^h \varphi_{2m}(z)dz; \qquad 0 < \theta < 1$$

or, since[4]

(18)
$$-\int_0^h \varphi_{2m}(z)dz = \frac{(-1)^{m+1}B_m h^{2m+1}}{(2m)!}$$

we shall have

(19)
$$R_m = \frac{(-1)^{m+1}B_m h^{2m}}{(2m)!}\sum_{x=a}^{b-h} f^{(2m)}(x+\theta h).$$

Whence, also

(20)
$$R_m = \Theta\frac{B_m h^{2m-1}}{(2m)!}(b-a)M;^{5} \qquad \begin{cases} -1 < \Theta < 1 \\ M \geq |f^{(2m)}(x)|; \quad a \leq x \leq b, \end{cases}$$

so that we reach in summary the following result:

"If $f(x)$ be any (real) function of the real variable x which together with its first $2m$ derivatives is continuous within the interval (a, b) we may write formula (16) in which, if M represents a value as great as the maximum value of $|f^{(2m)}(x)|$ within the same interval, the expression R_m satisfies relations (19) and (20)."

6. Other important forms for the remainder in the Maclaurin sum formula may be obtained when further hypotheses are placed upon $f(x)$. Thus, let us suppose in the first place that $f^{(2m)}(x)$ does not change sign between $x = a$ and $x = b$. By applying the first law of the mean for integrals we may then write

(21)
$$\begin{aligned} R_m &= \frac{-1}{h}\varphi_{2m}(\theta h)\int_0^h \sum_{x=a}^{b-h} f^{(2m)}(x+z)dz \\ &= \frac{-1}{h}\varphi_{2m}(\theta h)[f^{(2m-1)}(b) - f^{(2m-1)}(a)]; \qquad 0 < \theta < 1. \end{aligned}$$

Whence, by (*a*) and (*b*) of § 3,

(22)
$$R_m = \frac{-1}{h}\Theta\varphi_{2m}\left(\frac{h}{2}\right)[f^{(2m-1)}(b) - f^{(2m-1)}(a)]; \qquad 0 < \Theta < 1.$$

Moreover, from (8) and (13) we have

$$\varphi_{2m}\left(\frac{h}{2}\right) = h^{2m}\left[\frac{1}{(2m)!}\frac{1}{2^{2m}} - \frac{1}{2}\frac{1}{(2m-1)!}\frac{1}{2^{2m-1}} + \frac{B_1}{1\cdot 2}\frac{1}{(2m-2)!}\frac{1}{2^{2m-2}} - \frac{B_2}{4!}\frac{1}{(2m-4)!}\frac{1}{2^{2m-4}} + \cdots + (-1)^m\frac{B_{m-1}}{(2m-2)!}\frac{1}{2!}\frac{1}{2}\right]$$

and it is a demonstrable property of the Bernoulli numbers that the expression

[4] See MALMSTEN (*l. c.*), p. 64, 4.

[5] Due originally to POISSON. See *Mém. de l'Acad. des Sciences*, Vol. 6 (1823), p. 590.

here appearing in square brackets is equal to

$$(- 1)^m \frac{2^{2m} - 1}{2^{2m-1}} \frac{B_m}{(2m)!}.\text{[6]} \tag{23}$$

Whence, by adding and subtracting the term

$$\frac{(- 1)^{m+1} B_m h^{2m-1}}{(2m)!} [f^{(2m-1)}(b) - f^{(2m-1)}(a)]$$

in the second member of (16) we obtain the following result:

"If $f(x)$ be a (real) function of the real variable x which together with its first $2m$ derivatives is continuous within the interval (a, b) and if its $2m$th derivative does not change sign between the same limits, we may write

$$\sum_{x=a}^{b-h} f(x) = \frac{1}{h}\int_a^b f(x)dx - \tfrac{1}{2}[f(b) - f(a)] + \frac{B_1 h}{2!}[f'(b) - f'(a)] - \frac{B_2 h^3}{4!}$$

$$\times [f'''(b) - f'''(a)] + \cdots + \frac{(- 1)^{m+1} B_m h^{2m-1}}{(2m)!} [f^{(2m-1)}(b) - f^{(2m-1)}(a)] + R_m,$$

where

$$R_m = (- 1)^{m+1} \left(\Theta \frac{2^{2m} - 1}{2^{2m-1}} - 1\right) \frac{B_m h^{2m-1}}{(2m)!} [f^{(2m-1)}(b) - f^{(2m-1)}(a)]; \qquad 0 < \Theta < 1."$$

Since

$$0 < \Theta \frac{2^{2m} - 1}{2^{2m-1}} = 2\Theta \frac{2^{2m} - 1}{2^{2m}} < 2 \tag{24}$$

we see that under the hypotheses of the above result the series (1), even though it be *divergent*, may be used to compute the value of

$$\sum_{x=a}^{b-h} f(x) \tag{25}$$

with an error numerically less than the absolute value of the last term taken.

More generally, it appears in the same manner that we shall have the above result whenever $f(x), f'(x), f''(x), \cdots f^{(2m)}(x)$ are continuous within the interval (a, b), while the expression

$$\sum_{x=a}^{b-h} f^{(2m)}(x + z)$$

does not change sign between $z = 0$ and $z = h$.

7. Again, let us now suppose that neither of the expressions

$$\sum_{x=a}^{b-h} f^{(2m)}(x + z), \qquad \sum_{x=a}^{b-h} f^{(2m+2)}(x + z) \tag{26}$$

changes sign between $z = 0$ and $z = h$. Replacing m by $m + 1$ in (16), using

[6] See MALMSTEN (*l. c.*), p. 70.

therein the form for R_{m+1} determined by (19), and comparing the result with that of § 5 (in which m is left unaltered), we obtain

$$\frac{B_{m+1}h^{2m+2}}{(2m+2)!}\sum_{x=a}^{b-h} f^{(2m+2)}(x+\theta h) = -\left\{\theta\frac{2^{2m}-1}{2^{2m-1}}-1\right\}\frac{B_m h^{2m-1}}{(2m)!}[f^{(2m-1)}(b)-f^{(2m-1)}(a)].$$

But

$$f^{(2m-1)}(b) - f^{(2m-1)}(a) = \int_0^h \sum_{x=a}^{b-h} f^{(2m)}(x+z)dz.$$

Whence, upon recalling that B_m and B_{m+1} are both positive, we see that the expression

$$\theta\frac{2^{2m}-1}{2^{2m-1}} - 1$$

will be negative and numerically less than 1 in case expressions (26) are of the same sign between $z = 0$ and $z = h$, while it will be positive and no greater than

$$\frac{2^{2m}-1}{2^{2m-1}} - 1 = \frac{2^{2m-1}-1}{2^{2m-1}}$$

in case expressions (26) are of opposite sign throughout the same domain. Thus we reach the following result:

"Let $f(x)$ be a (real) function of the real variable x which together with its first $2m$ derivatives is continuous within the interval (a, b) and is such that neither of the expressions

$$\sum_{x=a}^{b-h} f^{(2m)}(x+z), \qquad \sum_{x=a}^{b-h} f^{(2m+2)}(x+z)$$

changes sign between $z = 0$ and $z = h$. Then, according as these expressions preserve the same or opposite signs for the indicated values of z, we may write

$$\begin{aligned}\sum_{x=a}^{b-h} f(x) = \frac{1}{h}\int_a^b f(x)dx - \tfrac{1}{2}[f(b)-f(a)] + \frac{B_1 h}{2!}[f'(b)-f'(a)] \\ - \frac{B_2 h^3}{4!}[f'''(b)-f'''(a)] + \cdots \\ + (-1)^m\frac{B_{m-1}h^{2m-3}}{(2m-2)!}[f^{(2m-3)}(b)-f^{(2m-3)}(a)] + R_m,\end{aligned} \tag{27}$$

where

$$R_m = (-1)^{m+1}\theta\frac{B_m h^{2m-1}}{(2m)!}[f^{(2m-1)}(b)-f^{(2m-1)}(a)]; \qquad 0<\theta<1$$

and

$$\begin{aligned}\sum_{x=a}^{b-h} f(x) = \frac{1}{h}\int_a^b f(x)dx - \tfrac{1}{2}[f(b)-f(a)] + \frac{B_1 h}{2!}[f'(b)-f'(a)] \\ - \frac{B_2 h^3}{4!}[f'''(b)-f'''(a)] + \cdots \\ + (-1)^{m+1}\frac{B_m h^{2m-1}}{(2m)!}[f^{(2m-1)}(b)-f^{(2m-1)}(a)] + R_m,\end{aligned} \tag{28}$$

where

$$R_m = (-1)^{m+1}\Theta \frac{2^{2m-1}-1}{2^{2m-1}} \frac{B_m h^{2m-1}}{(2m)!} [f^{(2m-1)}(b) - f^{(2m-1)}(a)]; \qquad 0 < \Theta < 1."$$

Formula (27) was first established by Jacobi[7] in 1834. Whenever the conditions for its use are satisfied it is seen that the sum of any number of terms in the series (1) (*convergent* or *divergent*) gives the value of (25) with an error having the same sign as that of the *first term neglected* and less numerically than the absolute value of that term. Formula (28) is due to Malmsten.[8] Whenever it may be used the sum of any number of terms in the series (1) gives the value of (25) with an error having the same sign as that of the *last term taken* and less numerically than the absolute value of that term.

8. Another important and well-known form for the remainder in the Maclaurin Sum-Formula may be obtained when the function $f(x)$ may be regarded as an *analytic* function of a complex variable.

To see this we recall in the first place that if $f(w)$ and $\varphi(w)$ are any two functions of the complex variable w ($w = x + iy$) both analytic and single-valued in the neighborhood of the point $w = a$ and of which the second has a zero of the first order at the same point, then we have the formula

$$(29) \qquad \int \frac{f(w)}{\varphi(w)} dw = \theta i \frac{f(a)}{\varphi'(a)} + \eta_\epsilon \qquad \begin{cases} \lim\limits_{\epsilon=0} \eta_\epsilon = 0; \\ \eta_\epsilon \equiv 0 \quad \textit{if} \quad \theta = 2\pi, \end{cases}$$

where the integration is taken in the positive sense along the arc of a circular sector of small radius ϵ and center at $w = a$ and whose angle is θ. In fact, this formula results directly from well-known principles in the theory of complex integrals upon observing that in the present instance we may develop the function $f(w)/\varphi(w)$ in the form

$$\frac{c_{-1}}{w-a} + p(w); \qquad c_{-1} = \frac{f(a)}{\varphi'(a)},$$

where $p(w)$ is analytic at the point $w = a$.

An immediate and useful corollary of (29) is as follows:

" If $f(w)$ and $\varphi(w)$ are any two functions of the complex variable w both of which are single-valued and analytic in a region A of the w-plane and of which the latter vanishes within A only at the points $w = \lambda_1, \lambda_2, \cdots \lambda_n$ which are zeros of the first order, and if C_n designate any contour lying within A and including the points $w = \lambda_1, \lambda_2, \cdots \lambda_n$, we shall have

$$(30) \qquad \frac{1}{2\pi i} \int_{C_n} \frac{f(w)}{\varphi(w)} dw = \sum_{n=1}^{n} \frac{f(\lambda_n)}{\varphi'(\lambda_n)},$$

where the indicated integration is performed in the positive sense."

[7] See *Journ. für Math.*, Vol. 13 (1834), p. 270.

[8] See Malmsten (*l. c.*), p. 72.

We proceed to apply formulas (29) and (30) to our present problem.[9] For this purpose let us take[10]

$$\varphi(w) = e^{(2\pi i/h)(w-a)} - 1 \tag{31}$$

and let us suppose $f(w)$ to be any function which is analytic throughout a vertical strip of the w-plane extending to an infinite distance both above and below the axis of reals and including the two real points $w = a$, $w = b$ $(b > a)$. For the contour C_n let us take that formed by the line $w = a + iy$ (the point $w = a$ excluded), by the line $w = b + iy$ ($w = b$ excluded) and by the lines $w = x \pm ij$ ($j = constant > 0$) together with small semicircles of radius $\epsilon < h$ about the points $w = a$, $w = b$, the former extending to the right and the latter to the left.

Since $\varphi(w)$ has zeros of the first order at the points $w = a + ph$; $p = 0$, 1, 2, $\cdots$, while at the same points $\varphi'(w) = 2\pi i/h$, we obtain as a result of (30)

$$h\sum_{x=a}^{b-h} f(x) = hf(a) + \int_{C_n} \frac{f(w)}{\varphi(w)}\,dw. \tag{32}$$

We proceed to study in further detail the complex integral here appearing.

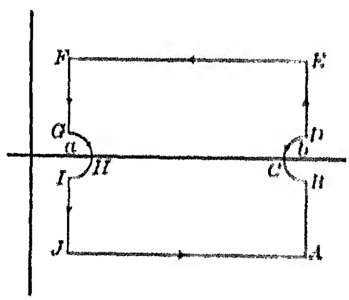

FIG. 1

First, the contribution coming from the side JA (see Fig. 1) is

$$I_1 = \int_a^b \frac{f(x - ij)}{\varphi(x - ij)}\,dx,$$

and since $\varphi(x - ij)$ becomes infinite when $j = +\infty$ like $e^{2\pi j/h}$, we have but to suppose that $f(w)$ satisfies the following supplementary condition:

$$\lim_{j=+\infty} f(x - ij)e^{-2\pi j/h} = 0; \qquad a \leqq x \leqq b \tag{33}$$

in order to have $I_1 = 0$ provided we take $j = \infty$. In particular, condition (33) will be satisfied whenever $|f(w)|$ remains less than a constant for all values of w within the strip already mentioned.

[9] See PETERSEN'S "Vorlesungen über Funktionstheorie" (Copenhagen, 1898), pp. 161–169.

[10] It is to be understood that the constants a, b, h have the meanings already introduced; viz., $b = a + nh$; $h > 0$, n = positive integer.

Secondly, let us consider the contribution coming from the portion $DEFG$. By writing

$$\frac{f(w)}{\varphi(w)} = \frac{\{\varphi(w)+1\}f(w)}{\varphi(w)} - f(w)$$

and observing that the integral of $f(w)$ over $DEFG$ is equal to that over $DCHG$, the contribution in question becomes

$$i\int_j^{\epsilon} \frac{\{\varphi(a+iy)+1\}f(a+iy)}{\varphi(a+iy)}dy + i\int_{\epsilon}^{j} \frac{\{\varphi(b+iy)+1\}f(b+iy)}{\varphi(b+iy)}dy$$
$$- \int_a^h \frac{\{\varphi(x+ij)+1\}f(x+ij)}{\varphi(x+ij)}dx + \int_{GHCD} f(w)dw.$$

Of the integrals here appearing we observe that the third may be neglected by taking $j = +\infty$ provided that $f(w)$ satisfy the following supplementary condition:

(34) $$\lim_{j=+\infty} f(x+ij)e^{-2\pi j/h} = 0; \qquad a \leqq x \leqq b.$$

Next, the contributions from the semicircular arcs BCD and GHI are equal respectively to $-(h/2)f(b)$ and $-(h/2)f(a)$ except for expressions that become infinitesimal with ϵ, as follows from (29).

We shall now assume not only the existence of (33) and (34) but that of the following stronger condition:

(35) $$\lim_{j=+\infty} f(x \pm ij)e^{[\eta-2\pi/h]j} = 0; \qquad a \leqq x \leqq b,$$

where η is an assignable positive quantity. If we then take account of the two remaining contributions, viz., those arising from the sides AB and IJ, we obtain in summary

$$h\sum_{x=a}^{b-h} f(x) = \int_{GHCD} f(w)dw - \frac{h}{2}[f(b)-f(a)] + i\int_{\epsilon}^{\infty} \frac{\{\varphi(b+iy)+1\}f(b+iy)}{\varphi(b+iy)}dy$$
$$+ i\int_{\infty}^{\epsilon} \frac{\{\varphi(a+iy)+1\}f(a+iy)}{\varphi(a+iy)}dy + i\int_{-\infty}^{-\epsilon} \frac{f(b+iy)}{\varphi(b+iy)}dy$$
$$+ i\int_{-\epsilon}^{-\infty} \frac{f(a+iy)}{\varphi(a+iy)}dy,$$

in which the various improper integrals have a meaning by virtue of (35).

Let us next allow ϵ to approach zero. Since

$$\varphi(a+iy) = \varphi(b+iy) = e^{-2\pi y/h} - 1,$$

we thus arrive at the equation

$$
\begin{aligned}
h\sum_{x=a}^{b-h} f(x) &= \int_a^b f(x)dx - \frac{h}{2}[f(b) - f(a)] \\
&\qquad + \frac{1}{i}\int_0^\infty \frac{[f(x+iy) - f(x-iy)]_{x=a}^{x=b}}{e^{2y\pi/h} - 1}\,dy.
\end{aligned} \tag{36}
$$

Moreover, the function $(1/i)\,[f(x+iy) - f(x-iy)]_{x=a}^{x=b}$, being real when y is real, may be expanded by Taylor's formula (with remainder) into the form

$$
\begin{aligned}
&2\left[\frac{1}{1!}f'(x)y - \frac{1}{3!}f'''(x)y^3 + \cdots + \frac{(-1)^{m-1}}{(2m-1)!}f^{(2m-1)}(x)y^{2m-1}\right]_{x=a}^{x=b} \\
&\qquad + \frac{(-1)^m y^{2m}}{(2m)!\,i}[f^{(2m)}(x+i\theta y) - f^{(2m)}(x-i\theta y)]_{x=a}^{x=b}; \qquad 0 < \theta < 1.
\end{aligned}
$$

Recalling finally that[11]

$$
\int_0^\infty \frac{y^{2p-1}}{e^{2\pi y/h} - 1}\,dy = h^{2p}\int_0^\infty \frac{y^{2p-1}}{e^{2\pi y} - 1}\,dy = \frac{h^{2p}}{4p}B_p; \qquad p = 1, 2, 3, \cdots,
$$

we reach the following result:

"If the function $f(w)$ is analytic throughout a vertical strip of the w complex plane extending to an infinite distance both above and below the axis of reals and including the real points $w = a$, $w = b$, and is furthermore such that

$$
\lim_{y=+\infty} f(x \pm iy)e^{(\eta - 2\pi/h)y} = 0; \qquad a \leqq x \leqq b,
$$

where η is some assignable positive quantity, we may write

$$
\begin{aligned}
\sum_{x=a}^{b-h} f(x) = \frac{1}{h}\int_a^b f(x)dx - \tfrac{1}{2}[f(b) - f(a)] &+ \frac{B_1 h}{2!}[f'(b) - f'(a)] \\
&- \frac{B_2 h^3}{4!}[f'''(b) - f'''(a)] + \cdots \\
&+ \frac{(-1)^{m-1}B_m}{(2m)!}[f^{(2m-1)}(b) - f^{(2m-1)}(a)] + R_m,
\end{aligned}
$$

where

$$
R_m = \frac{(-1)^m}{(2m)!\,ih}\int_0^\infty \frac{[f^{(2m)}(x+i\theta y) - f^{(2m)}(x-i\theta y)]_{x=a}^{x=b}}{e^{2y\pi/h} - 1}\,y^{2m}dy
$$

$$
\begin{cases} \theta = 1 & when \quad m = 0, \\ 0 < \theta < 1 & when \quad m = 1, 2, 3, \cdots \end{cases}
$$

Equation (36) with $h = 1$, was first given[12] by PLANA in 1820 and soon afterwards by ABEL.[13] The same result was obtained through the calculus of residues for the first time by KRONECKER[14] in 1889.

[11] See MALMSTEN (*l. c.*), p. 59.

[12] See *Mem. della Accad. delle sci. di Torino*, Vol. 25 (1820).

[13] See *Oeuvres complètes* (1881), Vol. 1, pp. 21–25. *Ibid.*, pp. 34–39.

[14] See *Journ. für Math.*, Vol. 105 (1889), p. 354.

9. We proceed to note certain theorems which follow from the preceding results and which will prove useful in the study of *divergent series* (Chap. II).

THEOREM I. *Let $f(x)$ be any (real) function of the real variable x which together with its first $2m$ derivatives is continuous throughout the infinite interval $x > a$. Also, let it be supposed that the following series is convergent:*[15]

$$\sum_{y=a}^{\infty} \int_a^1 f^{(2m)}(y+t)\varphi_{2m}(t)\,dt, \tag{37}$$

in which $\varphi_{2m}(t)$ represents the 2mth Bernoulli function. We may then write

$$\sum_{x=a}^{x-1} f(x) = C_m + \int_a^x f(x)dx - \tfrac{1}{2}f(x) + \frac{B_1}{2!}f'(x) - \frac{B_2}{4!}f'''(x) + \cdots$$
$$+ \frac{(-1)^m B_{m-1}}{(2m-2)!} f^{(2m-3)}(x) + \Omega_m(x),$$

where

$$\Omega_m(x) = \sum_{y=x}^{\infty} \int_0^1 f^{(2m)}(y+t)\varphi_{2m}(t)\,dt = \frac{(-1)^{m-1}B_m}{(2m)!}\sum_{y=x}^{\infty} f^{(2m)}(y+\theta_y); \quad \begin{cases} x \geqq a \\ 0 < \theta_y < 1, \end{cases} \tag{38}$$

and where C_m is a constant as regards x, defined by the equation

$$C_m = \tfrac{1}{2}f(a) - \frac{B_1}{2!}f'(a) + \frac{B_2}{4!}f'''(a) - \cdots + \frac{(-1)^{m-1}B_{m-1}}{(2m-2)!}f^{(2m-3)}(a) - \Omega_m(a).$$

In fact, the expression $\Omega_m(x)$ will exist for $x = a, a+1, a+2, \cdots$, and by the results of § 4 we shall have

$$\sum_{x=a}^{x-1} f(x) = \int_a^x f(x)dx - \tfrac{1}{2}[f(x) - f(a)] + \frac{B_1}{2!}[f'(x) - f'(a)] - \cdots$$
$$+ \frac{(-1)^m B_{m-1}}{(2m-2)!}[f^{(2m-3)}(x) - f^{(2m-3)}(a)] + R_m,$$

where

$$R_m = -\int_0^1 \sum_{y=a}^{x-1} f^{(2m)}(y+t)\varphi_{2m}(t)\,dt = \Omega_m(x) - \Omega_m(a).$$

But this result is coextensive with that indicated in the theorem. As to the second form there given for $\Omega_m(x)$, we observe that by virtue of statement (*a*) of § 3 we may apply the first law of the mean for integrals to each term of the series representing $\Omega_m(x)$, thus writing

$$\Omega_m(x) = \sum_{y=x}^{\infty} f^{(2m)}(y+\theta_y)\int_0^1 \varphi_{2m}(t)\,dt; \qquad 0 < \theta_y < 1,$$

[15] It will be understood that in this and the following two theorems y takes only the values $a, a+1, a+2, \cdots$.

vhich, upon using (18), becomes

$$\Omega_m(x) = \frac{(-1)^m B_m}{(2m)!} \sum_{y=x}^{\infty} f^{(2m)}(y + \theta_y). \tag{39}$$

We add that in case $\lim_{x=\infty} f^{(2p-1)}(x) = 0$; $p = 1, 2, 3, \cdots$, the constant C_m vill be independent of m as well as of x, for we shall then have

$$C_1 + \Omega_1(x) = C_m + \frac{B_1}{2!}f'(x) - \frac{B_2}{4!}f'''(x) + \cdots + \frac{(-1)^m B_{m-1}}{(2m-2)!} f^{(2m-3)}(x) + \Omega_m(x),$$

o that by placing $x = \infty$ and observing that $\lim \Omega_1(x) = \lim \Omega_m(x) = 0$, we btain $C_1 = C_m$.

THEOREM II. *Let $f(x)$ be any (real) function of the real variable x which ogether with its first $2m$ derivatives is continuous throughout the infinite interval $> a$. Also, let it be supposed that $f^{(2m)}(x)$ does not change sign within the same nterval and that $\lim_{x=\infty} f^{(2m-1)}(x) = 0$. We may then write*

$$\sum_{a}^{x-1} f(x) = C_m + \int_a^x f(x)dx - \tfrac{1}{2}f(x) + \frac{B_1}{2!}f'(x) - \frac{B_2}{4!}f'''(x) + \cdots$$
$$+ \frac{(-1)^{m-1}B_m}{(2m)!} f^{(2m-1)}(x) + \Omega_m(x),$$

here

$$\Omega_m(x) = \frac{(-1)^m B_m}{(2m)!} f^{(2m-1)}(x) + \sum_{y=x}^{\infty} \int_0^1 f^{(2m)}(y + t)\varphi_{2m}(t)dt$$
$$= \frac{B_m}{(2m)!}\Theta_m(x) f^{(2m-1)}(x); \qquad \begin{cases} x \geqq a \\ -1 \leqq \Theta_m(x) \leqq 1, \end{cases}$$

nd where C_m is a constant as regards x, defined by the equation

$$C_m = \tfrac{1}{2}f(a) - \frac{B_1}{2!}f'(a) + \frac{B_2}{4!}f'''(a) - \cdots + \frac{(-1)^m B_m}{(2m)!} f^{(2m-1)}(a) - \Omega_m(a).$$

To prove this theorem we first observe that, as a result of our hypotheses pon $f^{(2m)}(x)$, the terms of the expression

$$F_m(b, x) = \sum_{y=x}^{b-1} \int_0^1 f^{(2m)}(y + t)\varphi_{2m}(t)dt; \qquad b > x$$

ill all have the same sign, so that $F_m(b, x)$ is either an ever increasing or an ver decreasing function when b increases; also by treating $F_m(b, x)$ as we did he R_m of (17) by means of (21), (22) and (23) we obtain

$$F_m(b, x) = \frac{(-1)^m B_m}{(2m)!} \frac{2^{2m} - 1}{2^{2m-1}} \eta_m(b, x)\{f^{(2m-1)}(b) - f^{(2m-1)}(x)\};$$
$$0 < \eta_m(b, x) < 1.\text{[16]}$$

[16] The expression Θ appearing in § 5 is in general a function of m, a, b and h. Since, in the resent instance we have $h = 1$, $a = x$ we represent Θ by $\eta_m(b, x)$.

Whence, the expression

$$F_m(x) \equiv \lim_{b=\infty} F_m(b, x)$$

exists, and since by hypothesis

$$\lim_{b=\infty} f^{(2m-1)}(b) = 0,$$

we shall have

$$F_m(x) = \Omega_m(x) - \frac{(-1)^m B_m}{(2m)!} f^{(2m-1)}(x) = \frac{(-1)^{m+1} B_m}{(2m)!} \frac{2^{2m}-1}{2^{2m-1}} \eta_m(x) f^{(2m-1)}(x); \quad 0 \leqq \eta_m(x) \leqq 1$$

or

$$(40) \qquad \Omega_m(x) = \frac{(-1)^{m+1} B_m}{(2m)!} \left\{ \eta_m(x) \frac{2^{2m}-1}{2^{2m-1}} - 1 \right\} f^{(2m-1)}(x); \quad 0 \leqq \eta_m(x) \leqq 1.$$

Thus, $\Omega_m(x)$ exists and has the form indicated in the theorem.

Now, by equation (16) we shall have also

$$\sum_{x=a}^{x-1} f(x) = \int_a^x f(x)dx - \tfrac{1}{2}[f(x) - f(a)] + \frac{B_1}{2!}[f'(x) - f'(a)] - \cdots + \frac{(-1)^{m+1} B_m}{(2m)!} [f^{(2m-1)}(x) - f^{(2m-1)}(a)] + \Omega_m(x) - \Omega_m(a).$$

Thus, we reach the desired result. Again we note that C_m will be independent of m as well as of x whenever

$$\lim_{x=\infty} f^{(2p-1)}(x) = 0; \qquad p = 1, 2, 3, \cdots.$$

THEOREM III. *Let $f(x)$ be any (real) function of the real variable x which together with its first $2m + 2$ derivatives is continuous throughout the infinite interval $x > a$. Also, let it be supposed that $f^{(2m)}(x)$ and $f^{(2m+2)}(x)$ do not change sign within the same interval, while*

$$\lim_{x=\infty} f^{(2p-1)}(x) = 0; \qquad p = 1, 2, 3, \cdots.$$

Then, if $f^{(2m)}(x)$ and $f^{(2m+2)}(x)$ preserve the same sign $(x > a)$ we may write

$$\sum_{x=a}^{x-1} f(x) = C + \int_a^x f(x)dx - \tfrac{1}{2}f(x) + \frac{B_1}{2!}f'(x) - \frac{B_2}{4!}f'''(x) + \cdots + \frac{(-1)^m B_{m-1}}{(2m-2)!} f^{(2m-3)}(x) + \Omega_m(x),$$

where

$$\Omega_m(x) = \sum_{y=x}^{\infty} \int_0^1 f^{(2m)}(y+t)\varphi_{2m}(t)dt = (-1)^{m+1}\Theta_m(x) \frac{B_m}{(2m)!} f^{(2m-1)}(x); \quad \begin{cases} x \geqq a \\ 0 \leqq \Theta_m(x) \leqq 1, \end{cases}$$

and where C is a constant as regards both m and x, defined by the equation

$$C = \tfrac{1}{2}f(a) - \frac{B_1}{2!}f'(a) + \frac{B_2}{4!}f'''(a) - \cdots + \frac{(-1)^{m-1}B_{m-1}}{(2m-2)!}f^{(2m-3)}(a) - \Omega_m(a).$$[17]

On the other hand, if $f^{(2m)}(x)$ and $f^{(2m+2)}(x)$ preserve opposite signs ($x > a$) (other conditions remaining as before) we may write

$$\sum_{x=a}^{x-1} f(x) = C + \int_a^x f(x)dx - \tfrac{1}{2}f(x) + \frac{B_1}{2!}f'(x) - \frac{B_2}{4!}f'''(x) + \cdots$$
$$+ \frac{(-1)^{m+1}B_m}{(2m)!}f^{(2m-1)}(x) + \Omega_m(x),$$

where

$$\Omega_m(x) = \frac{(-1)^m B_m}{(2m)!}f^{(2m-1)}(x) + \sum_{y=x}^{\infty}\int_0^1 f^{(2m)}(y+t)\varphi_{2m}(t)dt$$
$$= (-1)^{m+1}\Theta_m(x)\frac{2^{2m-1}-1}{2^{2m-1}}\frac{B_m}{(2m)!}f^{(2m-1)}(x); \quad \begin{cases} x \geqq a \\ 0 \leqq \Theta_m(x) \leqq 1, \end{cases}$$

and where C is a constant as regards both m and x, defined by the equation

$$C = \tfrac{1}{2}f(a) - \frac{B_1}{2!}f'(a) + \frac{B_2}{4!}f'''(a) - \cdots + \frac{(-1)^m B_m}{(2m)!}f^{(2m-1)}(a) - \Omega_m(a).$$

For the proof of this theorem we first observe that the conditions for theorem II, and hence also those for theorem I, are here fulfilled both for $m = m$ and $m = m + 1$; also the conditions that C_m shall be independent of m. Upon applying theorem II with $\Omega_m(x)$ as given by (40) and comparing the result with that obtained by placing $m = m + 1$ in theorem I, we obtain

$$(41) \quad \frac{B_m}{(2m)!}\left\{\eta_m(x)\frac{2^{2m}-1}{2^{2m-1}} - 1\right\}f^{(2m-1)}(x) = \frac{B_{m+1}}{(2m+2)!}\sum_{y=x}^{\infty}f^{(2m+2)}(y+\theta_y).$$

Let us now write $f^{(2m-1)}(x)$ in the form

$$-\lim_{b=\infty}\int_0^1\sum_{y=x}^{b-1}f^{(2m)}(y+t)dt$$

and let $\Omega_m'(x)$ represent the expression $\Omega_m(x)$ of theorem II in the present discussion. Then, in case $f^{(2m)}(x)$ and $f^{(2m+2)}(x)$ preserve the same sign it follows from (41) that

$$(42) \qquad \Omega_m'(x) = \frac{(-1)^{m+1}B_m}{(2m)!}\psi_m(x)f^{(2m-1)}(x); \qquad -1 \leqq \psi_m(x) \leqq 0$$

and hence, for the first expression $\Omega_m(x)$ of the present theorem, we shall have

$$\Omega_m(x) = \Omega_m'(x) + \frac{(-1)^{m+1}B_m}{(2m)!}f^{(2m-1)}(x) = \frac{(-1)^{m+1}B_m}{(2m)!}\Theta_m(x)f^{(2m-1)}(x);$$
$$0 \leqq \Theta_m(x) \leqq 1$$

with which the first part of the theorem becomes established.

[17] Cf. MARKOFF (*l. c.*), pp. 131–133.

If, on the other hand, $f^{(2m)}(x)$ and $f^{(2m+2)}(x)$ preserve opposite signs ($x > a$) we shall have equation (42) in which

$$0 \leqq \psi_m(x) \leqq \frac{2^{2m}-1}{2^{2m-1}} - 1 = \frac{2^{2m-1}-1}{2^{2m-1}}$$

and thus the second part of the theorem becomes established, upon observing finally that we here have $\Omega_m(x) = \Omega_m'(x)$.

THEOREM IV. *Let $f(w)$ be any function of the complex variable $w = x + iy$ which is analytic throughout all portions of the w plane ($w = \infty$ excl.) for which $x \geqq a$. Also, let it be supposed that*

$$\lim_{y=+\infty} f(x \pm iy)e^{(\eta-2\pi)y} = 0; \qquad x \geqq a,$$

where η is some assignable positive quantity. We may then write

$$\sum_{x=a}^{x-1} f(x) = C_m + \int_a^x f(x)dx - \tfrac{1}{2}f(x) + \frac{B_1}{2!}f'(x) - \frac{B_2}{4!}f'''(x) + \cdots$$
$$+ \frac{(-1)^{m+1}B_m}{(2m)!}f^{(2m-1)}(x) + \Omega_m(x),$$

where

$$\Omega_m(x) = \frac{(-1)^m}{(2m)!\,i}\int_0^\infty \frac{f^{(2m)}(x+\theta_x iy) - f^{(2m)}(x-\theta_x iy)}{e^{2\pi y}-1}y^{2m}dy;$$
$$\begin{cases} \theta_x = 1 \text{ when } m = 0 \\ 0 < \theta_x < 1 \text{ when } m = 1, 2, 3, \cdots, \end{cases}$$

and where C_m is a constant as regards x, defined by the equation

$$C_m = \tfrac{1}{2}f(a) - \frac{B_1}{2!}f'(a) + \frac{B_2}{4!}f'''(a) - \cdots + \frac{(-1)^m B_m}{(2m)!}f^{(2m-1)}(a) - \Omega_m(a).$$

This theorem is, in fact, a direct consequence of the result stated in § 8, being obtained from it by placing $b = x$ and rearranging terms.

GENERALIZATION OF THE PRECEDING RESULTS[18]

10. The results given in § 4–7 and the first three theorems of § 9 require that the function $f(x)$ together with its first $2m$ derivatives shall be continuous throughout a certain specified interval. When this condition is not satisfied the same results and theorems no longer exist, at least in general. However, in cases in which $f(x)$ satisfies the indicated condition except at a finite

[18] For a derivation of the Maclaurin sum-formula from the standpoint of Fourier series, see POISSON (*l. c.*). A still different method may be found in BOOLE'S "Treatise on Finite Differences" (London, 1860), pp. 80–84. The formula has been generalized in various directions by BARNES; see *Quart. Journ. of Math.*, Vol. 35 (1903), pp. 175–188; *Trans. of Cambridge Philosophical Soc.*, Vol. 19 (1904), p. 325; *Proceedings of London Math. Soc.* (2), Vol. 3 (1905), pp. 253–272.

number of points (at which discontinuity or uncertainty may exist) we may still obtain certain noteworthy results.

In order to show this we first observe that if u and v be any two functions of the (real) variable x which together with their first derivatives are continuous throughout the interval $(\alpha, \alpha + h)$ except at the point $x = \beta$, we may write

$$(43)\quad \left(\int_{\alpha}^{\beta-\epsilon} + \int_{\beta+\epsilon}^{\alpha+h}\right) u dv = uv \Big]_{x=\alpha}^{x=\alpha+h} - uv \Big]_{x=\beta-\epsilon}^{x=\beta+\epsilon} - \left(\int_{\alpha}^{\beta-\epsilon} + \int_{\beta+\epsilon}^{\alpha+h}\right) v du$$

ϵ being an arbitrarily small positive quantity. This is, in fact, a direct consequence of the ordinary formula for integration by parts.[19]

In particular, if u_x be a function which together with its first $2m + 1$ derivatives $(u', u'', \cdots u^{(2m+1)})$ is continuous within the interval $(\alpha, \alpha + h)$ except at the point $x = \beta$ we may obtain by repeated use of (43) the following result (*cf.* (3)):

$$\Delta u_{\alpha}^{(k)} = u_{\alpha+h}^{(k)} - u_{\alpha}^{(k)} = \sum_{p=1}^{2m-k} \frac{h^p}{p!} u_{\alpha}^{(k+p)} + \left[\sum_{p=0}^{2m-k} \frac{(h-z)^p}{p!} u_{\alpha+z}^{(k+p)}\right]_{z=\beta-\alpha-\epsilon}^{z=\beta-\alpha+\epsilon}$$
$$+ \left(\int_0^{\beta-\alpha-\epsilon} + \int_{\beta-\alpha+\epsilon}^{h}\right) \frac{(h-z)^{2m-k}}{(2m-k)!} u_{\alpha+z}^{(2m+1)} dz.$$

Whence, if $H_0, H_1, \cdots H_{2m}$ be the constants defined by (5) we may write (*cf.* (6)):

$$hu_{\alpha}' = \sum_{k=0}^{2m-1} H_k h^k \Delta u_{\alpha}^{(k)} + \left[\sum_{k=0}^{2m-1} H_k h^k \sum_{p=0}^{2m-k} \frac{(h-z)^p}{p!} u_{\alpha+z}^{(k+p)}\right]_{z=\beta-\alpha-\epsilon}^{z=\beta-\alpha+\epsilon} + r_m(\alpha, h),$$

where

$$r_m(\alpha, h) = -\left(\int_0^{\beta-\alpha-\epsilon} + \int_{\beta-\alpha+\epsilon}^{h}\right) u_{\alpha+z}^{(2m+1)} \sum_{k=0}^{2m-1} H_k h^k \frac{(h-z)^{2m-k}}{(2m-k)!} dz.$$

Upon introducing the function $\varphi_{2m}(z)$ and making use of the relations (10) and (13) we thus obtain

$$(44)\quad \begin{aligned} hu_{\alpha}' &= \Delta u_{\alpha} - \frac{h}{2}\Delta u_{\alpha}' + \sum_{k=1}^{m-1} (-1)^{k-1} \frac{B_k h^{2k}}{(2k)!} \Delta u_{\alpha}^{(k)} \\ &\quad + \left[\sum_{k=0}^{2m-2} H_k h^k \sum_{p=0}^{2m-k} \frac{(h+\alpha-x)^p}{p!} u_{\beta+x}^{(k+p)}\right]_{x=-\epsilon}^{x=+\epsilon} + r_m(\alpha, h), \end{aligned}$$

where

$$r_m(\alpha, h) = -\left(\int_{\alpha}^{\beta-\epsilon} + \int_{\beta+\epsilon}^{\alpha+h}\right) u_x^{(2m+1)} \varphi_{2m}(x-\alpha) dx.$$

The case of especial interest for our present purpose is that in which u_x is taken in the following manner

[19] As usually stated (cf. GOURSAT, "Cours d'Analyse," Vol. 1 (1902), § 85) the formula requires that u and v with their first derivatives shall be continuous throughout the interval of integration.

$$u_x = \int_a^x f(x)dx \qquad when \qquad \alpha \leqq x \leqq \beta - \epsilon,$$

$$u_x = \left(\int_a^{\beta-\epsilon} + \int_{\beta+\epsilon}^x\right) f(x)dx \qquad when \qquad \beta + \epsilon \leqq x \leqq \alpha + h,$$

$f(x)$ being any function which together with its first $2m$ derivatives is continuous within the interval $(\alpha, \alpha + h)$ except at the point $x = \beta$. Such a function u_x together with its first $2m + 1$ derivatives will be continuous except at $x = \beta$.

Whence, applying (44), we may write[20]

$$(45) \qquad \begin{aligned} hf(\alpha) = {} & \left(\int_a^{\beta-\epsilon} + \int_{\beta+\epsilon}^{a+h}\right)[f(x) - f^{(2m)}(x)\varphi_{2m}(x-\alpha)]dx - \frac{h}{2}\Delta f(\alpha) \\ & + \sum_{k=1}^{m-1} (-1)^{k-1} \frac{B_k h^{2k}}{(2k)!} \Delta f^{(2k-1)}(\alpha) \\ & + \left[\sum_{k=0}^{2m-2} H_k h^k \sum_{p=0}^{2m-k} \frac{(h+\alpha-x)}{p!} f^{(k+p-1)}(\beta + x)\right]_{x=-\epsilon}^{x=+\epsilon}. \end{aligned}$$

Let us suppose lastly that the interval $(\alpha, \alpha + h)$ containing the point $x = \beta$ is part of a larger interval (a, b) throughout which (except at $x = \beta$) $f(x)$ satisfies the indicated conditions; also let us suppose that α is one of the quantities $a, a + h, a + 2h, \cdots, b - h$. If then we apply formula (16) to $f(x)$ when considered within the intervals (a, α), $(\alpha + h, b)$ and apply formula (45) to the same function when considered within the interval $(\alpha, \alpha + h)$ we obtain, after adding the three results and dividing by h,

$$(46) \qquad \begin{aligned} \sum_{q=0}^{n-1} f(a + qh) = \sum_{x=a}^{b-h} f(x) = {} & \frac{1}{h}\left(\int_a^{\beta-\epsilon} + \int_{\beta+\epsilon}^{b}\right) f(x)dx \\ & - \frac{1}{h}\left(\int_a^{\beta-\epsilon} + \int_{\beta+\epsilon}^{a+h}\right) f^{(2m)}(x)\varphi_{2m}(x-\alpha)dx \\ & + \left[\sum_{k=0}^{2m-2} H_k h^k \sum_{p=0}^{2m-k} \frac{(h+\alpha-x)^p}{p!} f^{(k+p-1)}(\beta + x)\right]_{x=-\epsilon}^{x=+\epsilon} \\ & - \tfrac{1}{2}[f(b) - f(a)] + \frac{B_1 h}{2!}[f'(b) - f'(a)] - \cdots \\ & + \frac{(-1)^m B_{m-1} h^{2m-3}}{(2m-2)!}[f^{(2m-3)}(b) - f^{(2m-3)}(a)] + R_m, \end{aligned}$$

where

$$R_m = -\frac{1}{h}\int_0^h \left(\sum_{x=a}^{a} + \sum_{x=a+h}^{b-h}\right) f^{(2m)}(x+z)\varphi_{2m}(z)dz.$$

By use of this formula instead of the earlier corresponding one (16) we arrive

[20] We note that $f^{(-1)}(\beta + x) = u_{\beta+x}$ and hence $f^{(-1)}(\beta + x)\Big]_{x=-\epsilon}^{x=\epsilon} = 0$.

at the desired theorems corresponding to the first three of § 9. Since these are long in statement though readily supplied we shall omit them.

Analogous results may evidently be obtained when $f(x)$ presents any (finite) number of exceptional points of the type just mentioned.

11. Again, the results stated in § 8 and the fourth theorem of § 9 require that $f(w)$ be analytic within a certain domain. If, on the other hand, this function presents singularities at a finite number of points within the domain, but otherwise satisfies the indicated conditions, we may readily make such alterations as are necessary to preserve correctness. For example, let us suppose that the function $f(w)$ of theorem IV satisfies the conditions there stated except at the point $w = \beta = p + iq$; $a < p < x$, $q < 0$. The theorem will then continue to hold true[21] provided that we subtract from the second member the residue r_β of the function

$$\frac{2\pi i f(w)}{\varphi(w)} = \frac{2\pi i f(w)}{e^{2\pi i(w-a)} - 1} \tag{47}$$

corresponding to the point $w = \beta$. However, if the exceptional point occurs at $w = \beta = p + iq$; $a < p < x$, $q > 0$, then (in view of the manner in which in § 8 the integral of $f(w)$ over the path $DEFG$ was transformed to one over the path $DCHG$) the theorem will continue true provided we subtract from the second member the expression r_β together with the residue r_β' of the function $2\pi i f(w)$ corresponding to the same point $w = \beta$.

Other cases are those in which a singular point occurs on either of the lines $w = a + iy$, $w = x + iy$ or at a *real* point $w = \beta < x$. If, in the last of these cases (which is the only one to which we shall refer later), the singular point is a pole of the first order the theorem is seen to continue as a result of (29) provided that the term

$$\int_a^x f(x)dx$$

be changed to

$$\lim_{\epsilon=0}\left(\int_a^{\beta-\epsilon} + \int_{\beta+\epsilon}^x\right) f(x)dx - r_\beta - \tfrac{1}{2}r_\beta',$$

where r_β, r_β' have the meanings already given.

SERIES OF STIRLING

12. As a preliminary application of the preceding general theorems to special functions $f(x)$ let us take $f(x) = \log x$, $a =$ *any real number* > 0. We are thereby led to certain well-known results respecting the series of Stirling.

The first part of theorem III may here be applied since we have

$$f^{(p)}(x) = \frac{(-1)^{p-1}(p-1)!}{x^p}; \qquad p = 1, 2, 3, \cdots.$$

[21] Cf. § 4.

Whence, upon observing that

$$\int_a^x \log x dx = [x \log x - x]_a^x = k_1 + x \log x - x; \qquad k_1 = const.$$

and that

$$\sum_{x=a}^{x-1} \log x = \log \Gamma(x) - \log \Gamma(a) = k_2 + \log \Gamma(x); \qquad k_2 = const.$$

we obtain

$$\log \Gamma(x) = K + (x - \tfrac{1}{2}) \log x - x + \frac{B_1}{1 \cdot 2} \frac{1}{x} - \frac{B_2}{3 \cdot 4} \frac{1}{x^3} + \frac{B_3}{5 \cdot 6} \frac{1}{x^5} - \cdots$$
$$+ \frac{(-1)^m B_{m-1}}{(2m-3)(2m-2)} \frac{1}{x^{2m-3}} + r_m(x),$$

where K is a constant as regards m and x and where

$$r_m(x) = \Theta_m(x) \frac{(-1)^{m+1} B_m}{(2m-1)(2m)} \frac{1}{x^{2m-1}}; \qquad 0 \leqq \Theta_m(x) \leqq 1.[22] \tag{48}$$

Moreover, by comparing the above results with the well-known formula[23]

$$\begin{aligned} \log \Gamma(x) = \tfrac{1}{2} \log 2\pi + (x - \tfrac{1}{2}) \log x - x \\ + \int_0^\infty \left(\frac{1}{1 - e^{-t}} - \frac{1}{t} - \frac{1}{2} \right) e^{-tx} \frac{dt}{t}, \end{aligned} \tag{49}$$

it follows (upon placing $x = \infty$) that $K = \frac{1}{2} \log 2\pi$.

Thus, we arrive at the series of Stirling (see Preface) and it appears from (48) that, though *divergent*, the series may be used to compute $\log \Gamma(x)$ with but slight error when x (real and positive) is large. In fact, the first term neglected is seen to constitute an upper limit to the error committed by breaking off the series at any one point. This fact was pointed out by CAUCHY[24] in 1843 through an independent investigation based upon formula (49), he also noting in this connection the possible value of *divergent series* in computation. CAUCHY's work was, however, confined to this one series and in this it appears that his results might have been obtained much more directly, as indicated in § 12, from the earlier general investigations of POISSON and JACOBI relative to the Maclaurin Sum-Formula.

We add that the value of the constant K may be obtained independently of formula (49) by use of the well-known formula of WALLIS expressing the value of $\pi/2$.[25]

[22] In the present case it may be shown that $0 < \Theta_m(x) < 1$. See MALMSTEN (*l. c.*), p. 75.

[23] Usually attributed to BINET.

[24] See *Comptes Rendus de l'Acad. des Sciences*, Vol. 17 (1843), pp. 370–376.

[25] See MARKOFF (*l. c.*), p. 134.

PRELIMINARY DISCUSSION OF ASYMPTOTIC SERIES

13. The formula of Stirling, by means of which the function

$$\log \Gamma(x) - (x - \tfrac{1}{2}) \log x + x$$

may be identified with a certain divergent power series in $1/x$, affords an illustration of an important class of developments known as *asymptotic series*. We proceed to give at this point a brief exposition of the general features of this subject, leaving its further development and applications for later chapters, especially chapters II and III.

Following POINCARÉ, we adopt the following definition:[26]

"A power series of the form

(50) $$a_0 + a_1\left(\frac{1}{x}\right) + a_2\left(\frac{1}{x}\right)^2 + \cdots; \qquad a_0, a_1, a_2, \cdots \textit{ constants}$$

is said to represent *asymptotically* the function $f(x)$ for large positive values of x whenever

(51) $$\lim_{x=+\infty} x^n [f(x) - (a_0 + a_1/x + a_2/x^2 + \cdots + a_n/x^n)] = 0; \qquad n = 0, 1, 2, 3, \cdots."$$ [27]

Thus, for a given value of n the difference between the function and the sum of the first $n + 1$ terms of its corresponding asymptotic series (in case one exists) vanishes to a higher order than the nth when $x = +\infty$, as would be the case in particular if the series were *convergent*. Symbolically, the above relation is expressed as follows:

(52) $$f(x) \sim a_0 + a_1/x + a_2/x^2 + \cdots.$$

Several general observations are here desirable. First, a given function $f(x)$ can be represented asymptotically in but one way. In fact, we have from (51)

(53) $$f(x) = a_0 + a_1/x + a_2/x^2 + \cdots + a_{n-1}/x^{n-1} + \frac{a_n + \epsilon_n(x)}{x^n}; \qquad \lim_{x=+\infty} \epsilon_n(x) = 0$$

[26] See *Acta Math.*, Vol. 8 (1886), p. 296.

[27] In this definition no restrictions are placed upon (50) as regards convergence or divergence. However, in the usual applications the series is divergent for all values (positive) of x, but as an instance in which the contrary is the case we have

$$e^{-1/x} \sim 0 + \frac{0}{x} + \frac{0}{x^2} + \frac{0}{x^3} + \cdots.$$

In the most important applications (cf. Chapters II and III) $f(x)$ is a function (either given explicitly or else determined implicitly as a solution of a linear differential or difference equation) capable of analytic continuation into the complex field, being in fact analytic throughout the finite plane with the exception of points (finite or infinite in number) situated upon a finite number of straight lines radiating from the origin and having the point $x = \infty$ as a non-polar singularity.

For further criticisms upon the definition of asymptotic series see THOMÉ, *Journ. für Math.*, Vol. 24 (1904), pp. 152–156; VAN VLECK, The Boston Colloquium Lectures (New York, Macmillan, 1905), pp. 77–85; WATSON, *Philosophical Trans.*, Vol. 211A (1911), pp. 279–313.

and in case we had also

$$f(x) = b_0 + b_1/x + b_2/x^2 + \cdots + b_{n-1}/x^{n-1} + \frac{b_n + \epsilon_n'(x)}{x^n}; \qquad \lim_{x\to+\infty} \epsilon_n'(x) = 0$$

we should have

$$(54)\qquad (a_0 - b_0) + (a_1 - b_1)\frac{1}{x} + (a_2 - b_2)\frac{1}{x^2} + \cdots + (a_{n-1} - b_{n-1})\frac{1}{x^{n-1}} + \frac{a_n - b_n + \epsilon_n(x) - \epsilon_n'(x)}{x^n} = 0.$$

Whence, $a_0 = b_0$, as results from the last equation by placing $x = +\infty$. Making use of this relation in (54), multiplying both members by x and proceeding as before, we obtain $a_1 = b_1, \cdots$, etc. The converse of the above statement is, however, not true as appears directly when we note that if $f(x)$ is represented asymptotically by (50) so also is, for example, the function $f(x) + e^{-x}$.[28]

Again, it is desirable for the sake of clearness to note that asymptotic series in general cannot be used for purposes of computation in the sense in which Stirling's series can be used to compute $\log \Gamma(x)$. In fact, no information is at hand respecting the error committed by stopping at any preassigned term.[29] There are, however, numerous and important asymptotic developments[30] which, like the series of Stirling, are derivable by use of the Maclaurin Sum-Formula and for such the limit of error may usually be fixed by means of the formulas then present for the remainder. But in all cases, the asymptotic development furnishes information as to the behavior of the function when x is very large. Thus, the expressions

$$a_0,\ a_0 + a_1/x,\ a_0 + a_1/x + a_2/x^2,\ \cdots,\ a_0 + a_1/x + a_2/x^2 + \cdots + a_m/x^m$$

constitute a series of successive approximations to the value of $f(x)$ provided that x is sufficiently large. Furthermore, we have

$$(55)\qquad \begin{aligned} &\lim_{x\to+\infty} f(x) = a_0 \\ &\lim_{x\to+\infty} x[f(x) - a_0] = a_1 \\ &\qquad\vdots \\ &\lim_{x\to+\infty} x^n[f(x) - a_0 - a_1/x - a_2/x^2 - \cdots - a_{n-1}/x^{n-1}] = a_n. \end{aligned}$$

Conversely, when the behavior of $f(x)$ for large positive values of x is known, the equations (55) serve to determine the coefficients $a_0, a_1, a_2, \cdots$ of the corresponding asymptotic development if one exists.

[28] By adopting a more limited definition of asymptotic series than that of POINCARÉ, WATSON has obtained a noteworthy theorem upon this question of uniqueness. See *Philosophical Trans.*, Vol. 211A (1911), p. 300.

[29] For noteworthy exceptional cases, see STIELTJES, *Annales de l'École Normale*, Vol. 13 (1886), pp. 201–262.

[30] This is true in general of the developments considered in Chapter II.

14. The following consequences of the definition (51) are especially noteworthy:[31]

"*If*

$$f(x) \sim a_0 + a_1/x + a_2/x^2 + \cdots,$$

$$\varphi(x) \sim b_0 + b_1/x + b_2/x^2 + \cdots$$

then

(*a*) $$f(x) \pm \varphi(x) \sim (a_0 \pm b_0) + \frac{a_1 \pm b_1}{x} + \frac{a_2 \pm b_2}{x^2} + \cdots;$$

(*b*) $$f(x) \cdot \varphi(x) \sim c_0 + c_1/x + c_2/x^2 + \cdots,$$

where $c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0$;

(*c*) $$f(x)/\varphi(x) \sim d_0 + d_1/x + d_2/x^2 + \cdots,$$

provided that $b_0 \neq 0$, *the coefficients* $d_0, d_1, d_2, \cdots$ *being determined by the equations*

$$\begin{cases} a_0 = b_0 d_0 \\ a_1 = b_1 d_0 + b_0 d_1 \\ \vdots \\ a_n = b_n d_0 + b_{n-1} d_1 + \cdots + b_0 d_n; \end{cases}$$

(*d*) $$\int_x^\infty f(x)dx \sim \frac{a_2}{x} + \frac{a_3}{2x^2} + \frac{a_4}{3x^3} + \cdots,$$

provided that $a_0 = a_1 = 0$."

In other words, asymptotic series are subject to the same laws of addition, subtraction, multiplication, division and term by term integration as convergent power series in $1/x$.

For the proof of (*a*) we have but to note that

$$f(x) = a_0 + a_1/x + a_2/x^2 + \cdots + a_{n-1}/x^{n-1} + \frac{a_n + \epsilon_n(x)}{x^n}; \qquad \lim_{x=+\infty} \epsilon_n(x) = 0,$$

$$\varphi(x) = b_0 + b_1/x + b_2/x^2 + \cdots + b_{n-1}/x^{n-1} + \frac{b_n + \epsilon_n'(x)}{x^n}; \qquad \lim_{x=+\infty} \epsilon_n'(x) = 0.$$

Thus, we may write

$$f(x) \pm \varphi(x) = (a_0 \pm b_0) + \cdots + (a_{n-1} \pm b_{n-1})\frac{1}{x^{n-1}} + \frac{(a_n \pm b_n) + \eta_n(x)}{x^n};$$

$$\lim_{x=+\infty} \eta_n(x) = 0.$$

As regards (*b*), let us indicate by $S_n(x)$, $T_n(x)$ and $\Sigma_n(x)$ respectively the sums of the first $n + 1$ terms of the three series in question. Placing for brevity $f(x) = f$, $\varphi(x) = \varphi$, $\epsilon_n(x) = \epsilon$, $\epsilon_n'(x) = \epsilon'$, $S_n(x) = S$, $T_n(x) = T$, $\Sigma_n(x) = \Sigma$, $\lim\limits_{x=+\infty} = \lim$, we shall have

[31] See POINCARÉ (*l. c.*), pp. 297–301.

(56) $$f = S + \frac{\epsilon}{x^n}, \qquad \varphi = T + \frac{\epsilon'}{x^n}; \qquad \lim \epsilon = \lim \epsilon' = 0$$

and

$$S \cdot T = \Sigma + \frac{P}{x^{2n}},$$

where P is a polynomial in x of degree no higher than the $(n - 1)$st.

Whence,

$$\left(f - \frac{\epsilon}{x^n}\right)\left(\varphi - \frac{\epsilon'}{x^n}\right) = \Sigma + \frac{P}{x^{2n}}$$

or

$$x^n[f \cdot \varphi - \Sigma] = f\epsilon' + \varphi\epsilon + (P - \epsilon\epsilon')\frac{1}{x^n}.$$

Now, $\lim f = a_0$; $\lim \varphi = b_0$ from which it follows that

$$\lim x^n[f \cdot \varphi - \Sigma] = 0.$$

For the proof of (c) let us use the same notation as above except that Σ shall represent the sum of the first $n + 1$ terms of the series in which the coefficients $d_0, d_1, d_2, \cdots$ appear. Then, using equations (56) we shall have

$$\frac{f}{\varphi} = \frac{S}{T} + \eta; \qquad \eta = \frac{\epsilon - \dfrac{S}{T}\epsilon'}{x^n T + \epsilon'}$$

and since $\lim S = a_0$, $\lim T = b_0 \neq 0$ it follows that $\lim x^n\eta = 0$.

Moreover,

$$\frac{S}{T} = \Sigma + \omega; \qquad \lim x^n\omega = 0.$$

Whence,

$$\lim x^n\left(\frac{f}{\varphi} - \Sigma\right) = \lim x^n(\eta + \omega) = 0.$$

The proof of (d) is readily supplied. We have from (53) when $a_0 = a_1 = 0$

$$\int_x^\infty f(x)dx = \frac{a_2}{x} + \frac{a_3}{2x^2} + \frac{a_4}{3x^3} + \cdots + \frac{a_{n-1}}{(n-2)x^{n-2}} + \frac{a_n + \eta_n(x)}{(n-1)x^{n-1}};$$

$$\eta_n(x) = x^{n-1}\int_x^\infty \frac{\epsilon_n(x)}{x^n}\,dx$$

and, since $\lim \epsilon_n(x) = 0$, we may say that corresponding to an arbitrarily small positive quantity δ there exists a constant x_δ such that $|\epsilon_n(x)| < \delta$; $x > x_\delta$.

Whence,

$$|\eta_n(x)| \leqq x^{n-1}\delta\int_x^\infty \frac{dx}{x^n} = \frac{\delta}{n-1}; \qquad x > x_\delta$$

so that $\lim \eta_n(x) = 0$.

In distinction, however, to the properties of convergent power series, the term by term derivative of the asymptotic development of $f(x)$ will not necessarily be the asymptotic development of $f'(x)$. This is most easily shown by an example. Thus,

(57) $$f(x) = e^{-x} \sin (e^x) \sim 0 + \frac{0}{x} + \frac{0}{x^2} + \cdots,^{22}$$

but since $f'(x) = -e^{-x} \sin (e^x) + \cos (e^x)$, the expression $\lim f'(x)$ is oscillatory so that not only does the term by term derivative of the series (57) fail to represent $f'(x)$ asymptotically, but $f'(x)$ permits of no such representation whatever.

However, if

$$f(x) \sim a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots$$

and if $f'(x)$ is known to be developable asymptotically, then

(58) $$f'(x) \sim -\frac{a_1}{x^2} - \frac{2a_2}{x^3} - \frac{3a_3}{x^4} - \cdots.$$

In fact, if $f'(x)$ were developable asymptotically in any other way than (58) it would follow from (d) of the above results that $f(x)$ was developable asymptotically in two different ways.

15. In addition to the properties (a), (b), (c) and (d) of § 14 we note also the following general result:

"*Let*

$$f(x) = a_0 + w(x); \qquad w(x) \sim \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots$$

and let $F(f)$ be a function of x through f which, when written in the form $F(a_0 + w)$, is developable as follows:

(59) $$F(a_0 + w) = F(a_0) + F'(a_0)w + \frac{F''(a_0)}{2!}w^2 + \cdots + \frac{F^{(n-1)}(a_0)}{(n-1)!}w^{n-1} + \frac{F^{(n)}(a_0) + \epsilon_n(w)}{n!}w^n; \qquad \lim_{w \to 0} \epsilon_n(w) = 0$$

(as happens in particular when $F(a_0 + w)$ is analytic at $w = 0$). Then we may write

$$F(f) \sim F(a_0) + \frac{p_1}{x} + \frac{p_2}{x^2} + \cdots + \frac{p_n}{x^n} + \cdots,$$

where $p_1, p_2, \cdots, p_n$ are the coefficients of the successive powers of $1/x$ obtained by substituting into (59) $\left(\text{exclusive of the term } \frac{\epsilon_n(w)}{n!}w^n\right)$ the first n terms of the given asymptotic development of $w(x)$."

[22] Cf. BROMWICH, "Infinite Series" (London, 1908), p. 334.

In fact, from (b) of § 14 we may write

$$F(a_0) + F'(a_0)w + \frac{F''(a_0)}{2!}w^2 + \cdots + \frac{F^{(n)}(a_0)}{n!} \sim F(a_0) + \frac{p_1}{x} + \frac{p_2}{x^2} + \cdots,$$

and hence (59) may be written in the form

$$F(a_0 + w) = F(a_0) + \frac{p_1}{x} + \frac{p_2}{x^2} + \cdots + \frac{p_n + \eta_n(x)}{x^n} + \frac{\epsilon_n(w)}{n!}w^n; \qquad \lim_{x=\infty} \eta_n(x) = 0.$$

If we now write $\epsilon_n(w)w^n$ in the form

$$\frac{1}{x^n}\epsilon_n(w)w^n x^n$$

and observe that

$$\lim_{x=\infty} \epsilon_n(w)w^n x^n = 0$$

we obtain the desired result.

16. We note in connection with the definition (51) that we have supposed x real and positive. More generally, $f(x)$ is said to be represented asymptotically by the series (50) throughout an infinite region T (usually a sector with center at $x = 0$) of the complex plane when, for all corresponding x values, the equation (51) exists in which $\lim_{|x|=\infty}$ is substituted for $\lim_{x=\infty}$. In the case frequently presented of a single-valued function $f(x)$ having an essential singularity at the point $x = \infty$, we note that the above mentioned region cannot completely surround the point $x = \infty$, since we should then have $\lim_{|x|=\infty} f(x) = a_0$ for *all* methods of increase of $|x|$, thus contradicting the hypothesis that the point $x = \infty$ is essentially singular.

Again, if $f(x)$ and the region T be given, we observe that the necessary and sufficient condition that $f(x)$ be developable asymptotically throughout T is that there exist a set of constants $a_0, a_1, a_2, \cdots, a_n, \cdots$ satisfying relations (55), it being understood that the values of x appearing in these relations are confined to T. In fact, if (55) exist we have (51) and conversely. The same relations (55), when employed as a sufficient test for the existence of an asymptotic development for $f(x)$ throughout T, are usually difficult to apply and hence of little value in practice, since $f(x)$ is not in general so given that it is possible to determine whether the indicated limits (representing $a_0, a_1, a_2, \cdots$) exist. A sufficient test which has a wider field of applicability is supplied by the following

THEOREM V.[33] *Let $f(x)$ be a function of the complex variable x analytic within and upon the boundary of a certain infinite region T of the x plane, the point $x = \infty$, however, being excluded. Also, let $\varphi(x) = f(1/x)$ and let T' be the region (having*

[33] Cf. FORD, *Bulletin Soc. Math. de France*, Vol. 39 (1911), p. 348. Line 13 should here read "le point $x = \infty$ toutefois étant exclu."

the point $x = 0$ upon its boundary) obtained from T by means of the transformation $x = 1/x'$. If, then, for values of x in T' the following limits exist:

$$\lim_{x=0} \varphi(x), \quad \lim_{x=0} \varphi'(x), \quad \lim_{x=0} \varphi''(x), \quad \cdots, \quad \lim_{x=0} \varphi^{(n)}(x), \quad \cdots$$

and are represented respectively by $\varphi(0)$, $\varphi'(0)$, $\cdots$, $\varphi^{(n)}(0)$, $\cdots$ (these values being assumed independent of the direction of approach of x to 0 in T') we may write for values of x in T

$$f(x) \sim a_0 + a_1(1/x) + a_2(1/x)^2 + \cdots + a_n(1/x)^n + \cdots$$

where

$$a_k = \frac{\varphi^{(k)}(0)}{k!}; \qquad k = 0, 1, 2, 3, \cdots, n, \cdots.$$

In order to prove this Theorem we shall begin by establishing the following Lemma in the general theory of functions:

Lemma I. "Let $\varphi(x)$ be a function of the complex variable x analytic within and upon the boundary of a certain region T' of the x-plane, exception being made, however, of the point $x = 0$ situated upon the boundary at which point $\varphi(x)$ may have any character. If, then, for values of x within T' the following $n + 2$ limits exist:

$$\lim_{x=0} \varphi(x), \quad \lim_{x=0} \varphi'(x), \quad \lim_{x=0} \varphi''(x), \quad \cdots, \quad \lim_{x=0} \varphi^{(n+1)}(x)$$

and are represented respectively by $\varphi(0)$, $\varphi'(0)$, $\varphi''(0)$, $\cdots$, $\varphi^{(n+1)}(0)$ (these values being assumed independent of the direction of approach of x to 0 in T') we may write for values of x in T'

$$\varphi(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + [a_n + r_n(x)]x^n; \qquad \lim_{x=0} r_n(x) = 0,$$

a_0, a_1, a_2, $\cdots$, a_n being constants determined by the equation

$$a_k = \frac{\varphi^{(k)}(0)}{k!} \; (k = 0, 1, 2, \cdots, n)."$$

In fact, under the above hypotheses we may write for any value of x in T'

$$\varphi(x) = \varphi(c) + \varphi'(c)(x - c) + \frac{\varphi''(c)}{2!}(x - c)^2 + \cdots + \frac{\varphi^{(n)}(c)}{n!}(x - c)^n + \frac{1}{n!}\int_c^x (x - t)^n \varphi^{(n+1)}(t)\,dt, \tag{60}$$

where c represents a fixed value of x taken within T' and arbitrarily near to 0 and where (at least if x and c are each taken sufficiently near to 0) the integration is understood to take place along the straight line joining the point $x = c$ to the point $x = 0$. The existence of (60) may be readily verified by performing an integration by parts n times upon the last term in the second member.[34]

[34] Cf. GOURSAT, "Cours d'Analyse," Vol. 1 (1902), § 86.

If in (60) we now allow c to approach the limit zero through values that lie within T' (x fixed), and if we introduce at the same time our hypotheses concerning the existence and meaning of $\varphi(0)$, $\varphi'(0)$, $\varphi''(0)$, $\cdots$, $\varphi^{(n)}(0)$, we obtain

$$\varphi(x) = \varphi(0) + \varphi'(0)x + \frac{\varphi''(0)}{2!}x^2 + \cdots + \frac{\varphi^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{\varphi^{(n)}(0) + r_n(x)}{n!}x^n, \tag{61}$$

where

$$r_n(x) = \int_0^x \left(\frac{x-t}{x}\right)^n \varphi^{(n+1)}(t)dt. \tag{62}$$

In order to complete the proof of the Lemma it thus remains but to show that with $r_n(x)$ defined as in (62) we shall have $\lim\limits_{x=0} r_n(x) = 0$ provided always that x remain in T'.

Now, for all values of t on the line of integration in (60) we have

$$\left|\frac{x-t}{x}\right| \leqq 1.$$

Moreover, it follows from our hypotheses that we may find a positive constant M (independent of x) such that for all values of x in T' we may write $|\varphi^{(n+1)}(x)| < M$. Whence, if we place $|x| = \rho$ we shall have for the given value of x

$$|r_n(x)| < M\int_0^\rho d\rho = M\rho$$

from which the desired result becomes evident.

Theorem I follows as an immediate consequence of the Lemma upon subjecting the function $f(x)$ and the region T mentioned in the theorem to the transformation $x = 1/x'$.

We note also that if, instead of having $f(x)$ defined throughout a complex region T, it is given as a function of a real variable x within the infinite interval $(a, +\infty)$, we may obtain in like manner the following Lemma and corresponding Theorem:

Lemma II. "Let $\varphi(x)$ be a function of the real variable x which, together with its first $n+1$ derivatives, is continuous within the interval $(0, b)$, the end point $x = 0$ being excluded. If, then, the limits $\varphi(+0)$, $\varphi'(+0)$, $\varphi''(+0)$, $\cdots$, $\varphi^{(n+1)}(+0)$ exist, we may write for values of x in $(0, b)$

$$\varphi(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + [a_n + r_n(x)]x^n; \qquad \lim_{x=+0} r_n(x) = 0,$$

$a_0, a_1, a_2, \cdots, a_n$ being constants determined by the equation

$$a_k = \frac{\varphi^{(k)}(+0)}{k!} \quad (k = 0, 1, 2, \cdots, n)."$$

THEOREM VI. *Let $f(x)$ be a function of the real variable x which, together with its derivatives of all orders is continuous throughout the infinite interval $(a, +\infty)$. If upon placing $\varphi(x) = f(1/x)$ the following limits exist:*

$$\varphi(+0), \quad \varphi'(+0), \quad \varphi''(+0), \quad \cdots, \quad \varphi^{(n)}(+0), \quad \cdots,$$

we may write for values of x in $(a, +\infty)$

$$f(x) \sim a_0 + a_1\left(\frac{1}{x}\right) + a_2\left(\frac{1}{x}\right)^2 + \cdots + a_n\left(\frac{1}{x}\right)^n + \cdots,$$

where

$$a_k = \frac{\varphi^{(k)}(+0)}{k!}; \qquad k = 0, 1, 2, 3, \cdots, n, \cdots.$$

17. We observe finally that the use of the symbol $\sim$ is frequently broadened as follows: "If f, φ and ψ are three functions of x such that in the sense of § 13 we have

$$\frac{f - \varphi}{\psi} \sim a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots,$$

the same relation may be written in the form

$$f \sim \varphi + a_0\psi + \frac{a_1\psi}{x} + \frac{a_2\psi}{x^2} + \cdots." \tag{63}$$

Thus we write when x is real and positive (cf. § 12)

$$\log \Gamma(x) \sim (x - \tfrac{1}{2}) \log x + x + \tfrac{1}{2} \log 2\pi + \frac{B_1}{1 \cdot 2}\frac{1}{x} - \frac{B_2}{3 \cdot 4}\frac{1}{x^3} + \cdots.$$

Relation (52) may furthermore be written in the simple form $f \sim a_0$, this being especially true in applications of the theory (such as the determination of $\lim_{x=+\infty} f(x)$) wherein the values of the coefficients $a_1, a_2, a_3, \cdots$ play no part. Likewise, relations of the form (63) may be written $f \sim \varphi + a_0\psi$.

CHAPTER II

THE DETERMINATION OF THE ASYMPTOTIC DEVELOPMENTS OF A GIVEN FUNCTION

18. Let $F(z)$ be a given function of the complex variable z defined throughout the finite z-plane and such that (*a*) the point $z = \infty$ is a non-polar singular point and (*b*) when $|z|$ is sufficiently large and *arg* z lies within a given sector Λ (center at $z = 0$) there exist two functions $f_\lambda(z)$ and $\varphi_\lambda(z)$ each defined throughout Λ and a set of constants $a_{0,\lambda}, a_{1,\lambda}, a_{2,\lambda}, \cdots a_{n,\lambda}, \cdots$ such that for values of z in Λ we have

$$F(z) = f_\lambda(z) + \varphi_\lambda(z)\left[a_{0,\lambda} + \frac{a_{1,\lambda}}{z} + \frac{a_{2,\lambda}}{z^2} + \cdots + \frac{a_{n,\lambda} + w_{\lambda,n}(z)}{z^n}\right];$$

$$\lim_{|z|=\infty} w_{\lambda,n}(z) = 0.$$

Then, according to the definition of § 13 and the remarks of § 17, we may write for the indicated values of z

$$F(z) \sim f_\lambda(z) + \varphi_\lambda(z)\left[a_{0,\lambda} + \frac{a_{1,\lambda}}{z} + \frac{a_{2,\lambda}}{z^2} + \cdots\right].$$

This form of asymptotic development is of frequent occurrence and of prime importance in analysis. The problem of determining for a given $F(z)$ and Λ the corresponding $f_\lambda(z)$, $\varphi_\lambda(z)$ and $a_{0,\lambda}, a_{1,\lambda}, a_{2,\lambda}, \cdots$ (assuming that they exist) is usually one of considerable difficulty and, when regarded in a general sense, is one for which but fragmentary results exist at the present time. The known determinations appear to be either those for special functions of importance in Mathematical Physics, such as Bessel's function $J_n(z)$,[1] or for certain types of integral functions, notably those defined by infinite products.[2]

In the present Chapter it is proposed to show how the general theorems of Chapter I may be used, at least in certain cases, to make the above indicated determinations. In doing this we shall merely consider certain *special* functions $F(z)$. No attempt will be made to obtain theorems of great generality, partly because of the difficulty of such an undertaking, but chiefly because of the belief that a few well-chosen illustrations suffice to adequately impart the spirit and possibilities of the method employed. In each of the functions $F(x)$ considered,

[1] See for example LOMMEL, "Studien über die Bessel'schen Functionen" (1868), § 17.

[2] See for example BARNES, *Philosophical Transactions*, Vol. 199A (1902), pp. 411–500; *ibid.*, Vol. 206A (1906), pp. 249–297. Each of these memoirs contains an extended bibliography of the subject. See also MATTSON, "Contributions à la Théorie des Fonctions entières" (Thèse), Upsala, 1905.

only the functions $f_\lambda(z)$, $\varphi_\lambda(z)$ and the first one of the constants $a_{n,\lambda}$ which is not equal to zero are determined, since these three determinations constitute what is essential to the study of the behavior of $F(z)$ for large values of $|z|$. The method, however, permits equally of the determination of any one of the coefficients $a_{n,\lambda}$.

The functions $F(z)$ considered fall into two classes: (a) those defined by infinite products and (b) those defined by infinite series. Under (a) we have eventually considered (§§ 24–28) the asymptotic behavior of the general integral function of order > 0—a problem to which considerable attention has been devoted in recent years[3] and in connection with which we have entered into considerable detail owing to the importance of this and other analogous considerations in the general theory of functions. Under (b) we have eventually considered (§§ 28, 29) the asymptotic behavior of functions defined by power (Maclaurin) series—a subject of evident importance owing to the essential rôle of such series in analysis. The treatment for the latter is brief and indeed but fragmentary, yet it is believed that the most important known results (aside from those which concern the solutions of linear differential or linear difference equations)[4] have been indicated.

The determination of the asymptotic character of functions defined in other ways than as infinite products or infinite series might well have been considered also in the present chapter, as likewise the corresponding problem for certain noteworthy *special* functions.[5] We have, however, limited ourselves in the manner indicated above, feeling that not all aspects of the subject could receive treatment within the limits of the chapter while those of the greatest permanence in the general theory of functions have been included, we believe, through the present selection.

19. *Example* 1. *To obtain asymptotic developments for the function*

(1)
$$F(z) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 + z^2}.$$

We here choose a function which, as a result of the well-known formula[6]

$$\frac{\tan z}{2z} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 \frac{\pi^2}{4} - z^2}$$

[3] See note at the bottom of page 44.

[4] See Chapter III.

[5] For miscellaneous investigations of this description, see BARNES, *Edinburgh Trans.*, Vol. 19 (1904), pp. 426–439; *Proceedings London Math. Soc.*, Vol. 3 (1905), pp. 273–295; *ibid.*, Vol. 5 (1907), pp. 59–116; *Transactions Cambridge Philosophical Soc.*, Vol. 20 (1907), pp. 253–279; *Quarterly Journ. of Math.*, Vol. 38 (1907), pp. 116–140; HARDY, *Quarterly Journ. of Math.*, Vol. 37 (1906), pp. 369–378; LITTLEWOOD, *Transactions Cambridge Philosophical Soc.*, Vol. 20 (1907), pp. 323–370.

[6] See, for example, Tannery's "Introduction à la Théorie des fonctions d'une variable" (Paris, 1886), § 117.

may be evaluated in the form

(2) $$F(z) = \frac{\pi}{4z}\frac{e^{\pi z}-1}{e^{\pi z}+1},$$

and this fact will enable us to check our subsequent results.

In order to obtain the asymptotic developments of $F(z)$ as defined by (1), let us place

$$f_z(w) = \frac{1}{(2w+1)^2+z^2}$$

and regard z as having any fixed value $z = p + iq$, $i = \sqrt{-1}$, lying in a sector (center at $z = 0$) situated in the *right* half of the z complex plane and having neither of its bounding lines coincident with the axis of pure imaginaries. Then $f_z(w)$, considered as a function of the complex variable $w = x + iy$, satisfies the conditions demanded by Theorem IV ($a = 0$) of Chapter I, except that in case $|q| > 1$ the same function will present a single pole of the first order at the right of the pure imaginary axis, this pole being situated at the point $w = \frac{1}{2}(-1 - iz)$ if $q > 1$ and at the point $w = \frac{1}{2}(-1 + iz)$ if $q < -1$.

Thus we may apply the theorem, subject to the remarks of § 11, in order to obtain an expression for the sum

$$\sum_{x=0}^{x-1} f_z(x); \qquad x > p.$$

We shall now distinguish between the following four cases: (a) $|q| < 1$, (b) $q > 1$, (c) $q < -1$, (d) $q = \pm 1$.

In (a) we may make direct application of the theorem. Taking $m = 0$, we thus obtain

(3) $$\sum_{x=0}^{x-1}\frac{1}{(2x+1)^2+z^2} = C_z + \int_0^x \frac{dx}{(2x+1)^2+z^2} - \frac{1}{2}\cdot\frac{1}{(2x+1)^2+z^2} + \Omega_z(x),$$

where

(4) $$\Omega_z(x) = -i\int_0^\infty \frac{f_z(x+iy) - f_z(x-iy)}{e^{2\pi y}-1}\,dy$$

and

(5) $$C_z = \frac{1}{2(1+z^2)} - \Omega_z(0).$$

In these results let us now allow x to increase indefinitely, observing that

$$\int_0^\infty \frac{dx}{(2x+1)^2+z^2} = \frac{1}{2z}\left[\arc\tan\frac{2x+1}{z}\right]_{x=0}^{x=\infty} = \frac{\pi}{4z} - \frac{1}{2z}\arc\tan\frac{1}{z}$$

and that $\lim_{x\to\infty}\Omega_z(x) = 0$. We obtain

(6)
$$F(z) = \frac{\pi}{4z} + \frac{1}{2(1+z^2)} - \frac{1}{2z}\operatorname{arc\,tan}\frac{1}{z} + i\int_0^\infty \left[\frac{1}{(1+2iy)^2+z^2} - \frac{1}{(1-2iy)^2+z^2}\right]\frac{dy}{e^{2\pi y}-1}.$$

Upon developing the various terms of the second member in ascending powers of $1/z$, we thus reach (Theorem V, Chapter I) the relation

(7)
$$F(z) \sim \frac{\pi}{4z} + \frac{a_2}{z^2} + \frac{a_4}{z^4} + \frac{a_6}{z^6} + \cdots,$$

in which the coefficients $a_2, a_4, a_6, \cdots$ may be evaluated to any desired point.[7]

In case (b), equation (3) and hence (6) also, will continue to hold true according to § 11 provided that we subtract from its second member the residue of the function

(8)
$$\frac{2\pi i}{[(2w+1)^2+z^2][e^{2\pi i w}-1]}$$

at the point $w = -\frac{1}{2}(1+iz)$ which (residue) is readily found (cf. (30), Chapter I) to be $\pi/2z(e^{\pi z}+1)$. Since, for values of z within the proposed sector, this function is developable asymptotically in the form (30) of Chapter I with

$$a_0 = a_1 = a_2 = \cdots = 0,$$

it follows that relation (7) holds true also in case (b).

Similarly in case (c) we have equation (6) except (cf. § 11) that we must now subtract from its second member the residue of (8) at $w = -\frac{1}{2}(1-iz)$ and also that of the function $2\pi i f_z(w)$ at the same point; *i. e.*, we must subtract the expression

$$-\frac{\pi}{2z(e^{-\pi z}+1)} + \frac{\pi}{2z} = \frac{\pi}{2z(e^{\pi z}+1)}.$$

Thus, as in case (c) we see that relation (7) again holds true.

Moreover, the same relation continues in case (d) as appears by writing $F(z)$ in the form

$$F(z) = \frac{1}{1+z^2} + \sum_{n=1}^{\infty}\frac{1}{(2n+1)^2+z^2}$$

and applying the method of case (a) to the summation here appearing, also recalling that in one and the same region there can exist but one asymptotic development for a given function.

Similarly, if we note the effect in (4) of supposing the real part of z to be *negative*, we find that when z is situated in a sector lying within the *left* half of

[7] It may be noted that by using a sufficiently large value of m in applying Theorem IV (Chap. I) we may obtain any one of these coefficients in a relatively simple form involving the Bernoulli numbers.

the plane, relation (6) continues to exist provided that the term $\pi/4z$ be replaced by $-\pi/4z$.

Thus in summary we may say that *throughout any sector (vertex at $z = 0$) of the z plane which does not contain portions of the pure imaginary axis, the function $F(z)$ defined by (1) may be developed asymptotically in the form*

$$F(z) \sim \frac{\pm\pi}{4z} + \frac{a_2}{z^2} + \frac{a_4}{z^4} + \cdots,$$

wherein the upper or lower sign is to be taken according as we are dealing with a sector in which the real part of z is positive or negative.

This result, which is at once seen to be consistent with the known relation (2), illustrates in simple manner the way in which asymptotic developments for a given function may be ascertained, at least in some cases, by means of the general theorems of Chapter I. This will be further illustrated in what follows, wherein we shall eventually consider cases of much greater generality.[8]

20. In § 19 we have considered asymptotic developments of $F(z)$ (cf. (1)) which are valid in sectors situated in the *right* or *left* halves of the z complex plane. We proceed to show how the same method may yield analogous developments holding for the *upper* and *lower* halves of the plane, exception being made naturally of those (pure imaginary) points corresponding to the values

$$z = \pm(2n+1)i; \qquad n = 0, 1, 2, \cdots$$

at which $F(z)$ becomes infinite.

For this let us consider the function

$$\Phi(z) = F(iz) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 - z^2}.$$

Regarding z at first as *real*, we place

$$\varphi_z(w) = \frac{1}{(2w+1)^2 - z^2}$$

and again undertake to apply Theorem IV with $m = 0$. This can be done only in case $\varphi_z(w)$ is analytic in w throughout the right half of the w plane. How-

[8] In the special instance before us it may be shown that $a_2 = a_4 = a_6 = \cdots = 0$. In fact if we substitute in (7) the form for $F(z)$ given by (2) we obtain

$$\frac{\pi}{4z}\left[\frac{e^{\pi z}-1}{e^{\pi z}+1} \mp 1\right] = \frac{\mp\pi}{2z}\left[\frac{1}{1+e^{\pm\pi z}}\right] \sim \frac{a_2}{z^2} + \frac{a_4}{z^4} + \cdots,$$

where the upper or lower sign is to be taken according as the real part of z is positive or negative, and this relation is seen to be true when $a_2 = a_4 = a_6 = \cdots = 0$. It is to be noted, however, that *in general* if a function is defined by a series of the type of (1) (cf. (12)) no formula analogous to (2) is at hand. The indicated method for determining the asymptotic development of the function, however, remains the same, thus leading to coefficients $a_0, a_1, a_2, \cdots$, which are in general not all equal to zero.

ever, we are concerned with large values of $|z|$, and whenever $|z| > 1$ it is evident that $\varphi_z(w)$ will have a pole of the first order within the indicated region at the point $w = (z-1)/2$ or $w = -(z+1)/2$ according as z is positive or negative.

Let us first consider that z is *positive*. We proceed to apply the theorem, subject to the remarks of § 11.

Since the residues r_β, r_β' of the functions

$$2\pi i \varphi_z(w), \qquad 2\pi i \frac{\varphi_z(w)}{e^{2\pi i w} - 1}$$

at $w = \beta = \frac{1}{2}(z-1)$ are respectively

$$-\frac{\pi}{2iz}, \qquad \frac{\pi}{2iz(e^{\pi i z} + 1)}$$

o that

$$-r_\beta - \tfrac{1}{2} r_\beta' = \frac{\pi}{4iz} \frac{e^{\pi i z} - 1}{e^{\pi i z} + 1}$$

we may write (at least when $x > \frac{1}{2}(z-1)$)

$$\sum_{x=0}^{x-1} \frac{1}{(2x+1)^2 - z^2} = C_z + \frac{\pi}{4iz} \frac{e^{\pi i z} - 1}{e^{\pi i z} + 1}$$

9)

$$+ \lim_{\epsilon = 0} \left(\int_0^{\frac{1}{2}(z-1)-\epsilon} + \int_{\frac{1}{2}(z-1)+\epsilon}^{x} \right) \frac{dx}{(2x+1)^2 - z^2} - \frac{1}{2} \frac{1}{(2x+1)^2 - z^2} + \Omega_z(x)$$

n which C_z and $\Omega_z(x)$ are obtained by changing z^2 to $-z^2$ in (4) and (5).

But from elementary considerations, the third term in (9) reduces to

$$\frac{1}{4z} \log \frac{(z+1)(2x+1-z)}{(z-1)(2x+1+z)}.$$

Whence, upon allowing x to increase indefinitely we obtain ($z > 1$)

$$\Phi(z) = \frac{\pi}{4iz} \frac{e^{\pi i z} - 1}{e^{\pi i z} + 1} + \frac{1}{2(1 - z^2)} + \frac{1}{4z} \log \frac{z+1}{z-1}$$

.0)

$$-\frac{1}{i} \int_0^\infty \left[\frac{1}{(1 + 2iy)^2 - z^2} - \frac{1}{(1 - 2iy)^2 - z^2} \right] \frac{dy}{e^{2\pi y} - 1}$$

nd hence (Theorem V, Chapter I)

1)
$$\Phi(z) \sim \frac{\pi}{4iz} \frac{e^{\pi i z} - 1}{e^{\pi i z} + 1} + \frac{b_2}{z^2} + \frac{b_4}{z^4} + \cdots,$$

here $b_2, b_4, \cdots$ are determinate constants.

This result may now be generalized to all values of z belonging to a sector S enter at $z = 0$) lying in the right half of the z plane, exception being made, as ready indicated, of the points $z = 2n + 1$; $n = 0, 1, 2, \cdots$. In fact, we have it to suppose $|z| > 1$ to have in (10) two expressions equal for positive values

of z and each analytic throughout S and hence equal for all values of z in the same region.[9] Moreover, the last term in the second member (like the two preceding) is readily seen to be developable in ascending powers of $1/z^2$, thus leading to a series which, in the sense of § 13, represents the same term asymptotically for all values of z in S.

Likewise, the same relation (11) is found to hold true for a corresponding sector in the *left* half of the plane, exception being made of the points

$$z = -(2n+1); \qquad n = 0, 1, 2, \cdots$$

so that, having replaced z by $-iz$, we may say in summary that *throughout any sector (vertex at $z = 0$) of the z plane which does not contain portions of the real axis, the function $F(z)$ defined by* (1) *may be developed asymptotically in the form*

$$F(z) \sim \frac{\pi}{4z}\frac{e^{\pi z}-1}{e^{\pi z}+1} + \frac{b_2}{z^2} + \frac{b_4}{z^4} + \cdots.$$

This result is again seen to be consistent with the known relation (2).[10]

21. *Generalization of Example* 1. The method above illustrated for determining asymptotic developments is in general applicable to functions $F(z)$ defined by series of the form

$$F(z) = \sum_{n=0}^{\infty} \frac{\mu(n)}{\lambda(n) + p(z)}$$

where $p(z)$ is an integral function of z and where $\lambda(n)$, $\mu(n)$ are functions of n such that Theorem IV, subject to the remarks of § 11, may be applied to the expression

$$f_z(w) = \frac{\mu(w)}{\lambda(w) + p(z)}$$

in order to find for a given value of z the sum

$$\sum_{x=0}^{x-1} f_z(x).$$

We observe in particular that by taking $p(z) = z^q$ ($q = integer \geqq 1$) the expression $F(z)$ (or the sum of a number of such expressions) comes to include a wide variety of functions having radial clusters of polar singularities in the neighborhood of the point $z = \infty$—a characteristic common to many of the more important functions of analysis.

In cases where $f_z(w)$ cannot be considered as a function of the complex

[9] It may be remarked that the last term in the second member of (10) is analytic throughout S ($|z| > 1$) because the improper integral involved converges uniformly for values of z in any sub-region S' of S whose boundary does not touch the boundary of S. (Cf. OSGOOD, "Encyklopädie der math. Wiss.," II, 2, § 6.)

[10] In view of the same relation it appears from (11) that in the present simple case we have $b_2 = b_4 = b_6 = \cdots = 0$ and that the symbol $\sim$ may be changed to $=$. Cf. note 8, p. 35.

ariable $w = x + iy$ but is continuous in the real variable x we may frequently etermine the desired developments by use of Theorems I, II or III of Chapter I subject possibly to the remarks of § 10). The manner in which Theorem I may e thus used will be shown in the following example wherein an important ype of function $F(z)$ different from that of § 19 is taken.

22. *Example 2. To obtain asymptotic developments for the function*

$$F(z) = \prod_{n=1}^{\infty}\left[1 + \frac{z^2}{n^2}\right]. \tag{12}$$

As in example 1, this function may be evaluated beforehand and takes the orm

$$F(z) = \frac{e^{\pi z} - e^{-\pi z}}{2\pi z},[11] \tag{13}$$

hus furnishing a check upon our subsequent results.

We begin by writing

$$\log F(z) = \sum_{n=1}^{\infty}\log\left[1 + \frac{z^2}{n^2}\right] = \lim_{x=\infty}\left[\sum_{x=1}^{x-1}\log(x^2 + z^2) - 2\sum_{x=1}^{x-1}\log x\right]. \tag{14}$$

From § 12 we have

$$\begin{aligned} -2\sum_{x=1}^{x-1}\log x = -2\log\Gamma(x) = -\log 2\pi - 2(x - \tfrac{1}{2})\log x \\ + 2x + \omega_1(x); \qquad \lim_{x=+\infty}\omega_1(x) = 0. \end{aligned} \tag{15}$$

Ve proceed to apply Theorem I (Chap. I) with $m = 1$ to the first summation the last member of (14), taking for this purpose $f(x) = \log(x^2 + z^2)$ and ipposing for the present that z is *real* but different from zero. The theorem ay be applied since the series (37) (Chap. I) becomes

$$\Omega_z(x) = \sum_{y=x}^{\infty}\int_0^1\left\{\frac{d^2}{dx^2}\log(x^2 + z^2)\right\}_{x=y+t}\varphi_2(t)dt,$$

hich, as in (38), may be written in the form

$$-\frac{B_1}{2}\sum_{y=x}^{\infty}\left\{\frac{d^2}{dx^2}\log(x^2 + z^2)\right\}_{x=y+\theta_y}; \qquad 0 < \theta_y < 1$$

d is therefore convergent.

Thus we have

$$\begin{aligned} \sum_{x=1}^{x-1}\log(x^2 + z^2) = \tfrac{1}{2}\log(1 + z^2) + \Omega_z(1) \\ + \int_1^x \log(x^2 + z^2)dx - \tfrac{1}{2}\log(x^2 + z^2) + \Omega_z(x). \end{aligned} \tag{16}$$

[11] See, for example, TANNERY, *l. c.*, § 121.

Moreover,

$$\int^x \log (x^2 + z^2)dx = x \log (x^2 + z^2) - 2x + 2z \text{ arc tan } \frac{x}{z}$$

so that by combining relations (14), (15) and (16) we obtain

$$\log F(z) = -\log 2\pi - \tfrac{1}{2} \log (1 + z^2) - 2z \text{ arc tan } \frac{1}{z} + 2 - \Omega_z(1)$$
$$+ \lim_{x \to \infty} \left[(x - \tfrac{1}{2}) \log \left(1 + \frac{z^2}{x^2}\right) + 2z \text{ arc tan } \frac{x}{z} + \omega_1(x) + \Omega_z(x) \right].$$

But

$$\lim_{x \to \infty} (x - \tfrac{1}{2}) \log \left(1 + \frac{z^2}{x^2}\right) = 0; \qquad \lim_{x \to \infty} \omega_1(x) = 0; \qquad \lim_{x \to \infty} \Omega_z(x) = 0,$$

and, supposing at first that z is *positive*, we shall have $\lim\limits_{x \to \infty} 2z \text{ arc tan } (x/z) = \pi z$.

Therefore, we may write (z *real* > 0)

$$\text{(17)} \quad \log F(z) = -\log 2\pi z + \pi z - \tfrac{1}{2} \log \left(1 + \frac{1}{z^2}\right) + 2\left(1 - z \text{ arc tan } \frac{1}{z}\right) - \Omega_z(1).$$

On the other hand, if z is *negative* we obtain

$$\text{(18)} \quad \log F(z) = -\log (-2\pi z) - \pi z - \tfrac{1}{2} \log \left(1 + \frac{1}{z^2}\right) + 2\left(1 - z \text{ arc tan } \frac{1}{z}\right) - \Omega_z(1).$$

We now observe that the expression $\Omega_z(1)$ is a function of z which is single valued and analytic in any region whose boundary does not cross the axis of pure imaginaries. Whence, within any region Λ_1 situated in the *right* half of the z plane, equation (17) may be used, while similar remarks apply to equation (18) for values of z pertaining to any region Λ_2 in the *left* half of the plane. Moreover, if the boundaries of Λ_1 and Λ_2 are not tangent to the pure imaginary axis at ∞, the function $\Omega_z(1)$ vanishes like $1/z^2$ when $|z| = \infty$ in Λ_1 (or Λ_2) and is developable asymptotically by Theorem V, Chapter I, in powers of $1/z^2$ within this region. It therefore remains but to apply the result stated in § 15 in order to say that *throughout any sector (vertex at $z = 0$) of the z plane which does not contain portions of the pure imaginary axis, the function $F(z)$ defined by* (12) *may be developed asymptotically in the form*

$$F(z) \sim \frac{\pm e^{\pm \pi z}}{2\pi z} \left[1 + \frac{a_2}{z^2} + \frac{a_4}{z^4} + \cdots\right],$$

wherein the upper or lower sign is to be taken according as we have a sector in which the real part of z is positive or negative.

This result is at once seen to be consistent with the known relation (13).[12]

23. We proceed to show how asymptotic developments for the $F(z)$ of § 22 may be obtained which will be valid in sectors that may include the pure imaginary axis. For convenience we shall convert this problem into the following: "To determine asymptotic developments for the function

$$F(iz) = \Phi(z) = \prod_{n=1}^{\infty}\left[1 - \frac{z^2}{n^2}\right], \tag{19}$$

which shall hold good throughout certain sectors that include the real z axis."

Considering at first that z has a fixed, positive, non-integral value > 1, we proceed (cf. (14)) to study the expression

$$H(z) = \lim_{x=\infty}\left[\sum_{x=1}^{x-1}\log(x^2 - z^2) - 2\sum_{x=1}^{x-1}\log x\right], \tag{20}$$

in which we agree to write $\log(x^2 - z^2) = \log(z^2 - x^2) + \pi i$ whenever $x < z$. Then $e^{H(z)} = \Phi(z)$.

In order to obtain a form analogous to (16) for the first sum here appearing, let us place $\varphi_z(w) = \log(w + z) + \log(w - z)$, in which it is understood that the function $\log(w - z)$, considered as a function of the complex variable w, is rendered single valued throughout the right half of the w-plane by means of a cut extending from the point $w = z$ vertically downward to the point $w = \infty$.

We shall then have

$$\sum_{x=1}^{x-1}\log(x^2 - z^2) = \sum_{x=1}^{x-1}\varphi_z(x) = \sum_{x=1}^{x-1}\log(x + z) + \sum_{x=1}^{x-1}\log(x - z) \tag{21}$$

and we may at once apply Theorem IV ($m = 0$) of Chap. I to the first sum in the last member, thus writing

$$\begin{aligned}\sum_{x=1}^{x-1}\log(x + z) &= \tfrac{1}{2}\log(1 + z) - \Omega_z(1)\\ &\quad + \int_1^x \log(x + z)dx - \tfrac{1}{2}\log(x + z) + \Omega_z(x),\end{aligned} \tag{22}$$

where

$$\Omega_z(x) = -i\int_0^{\infty}\log\left(\frac{x + iy + z}{x - iy + z}\right)\frac{dy}{e^{2\pi y} - 1}. \tag{23}$$

The second sum, however, can not be treated in the same manner owing to the presence of an essential singularity of the function $\log(w - z)$ at the point $w = z$. In this case a related method yields the desired result as we shall now show.

[12] By means of (13) we may show (cf. note 8, p. 35) that in the present instance

$$a_2 = a_4 = a_6 = \cdots = 0.$$

Let us take for this purpose the integral of the function

$$\frac{f_z(w)}{\varphi(w)}; \qquad f_z(w) = \log (w - z), \qquad \varphi(w) = e^{2\pi i w} - 1$$

with respect to the complex variable w from the point $w = z - ij$ (j = any real, positive value, arbitrarily large) situated on the *right* side of the above mentioned cut, around the open contour $ABCDCEFGFHI$ indicated in the following figure, the integration terminating at the same point on the *left* side of the cut, and it being understood that the two closed loops CD and FG include

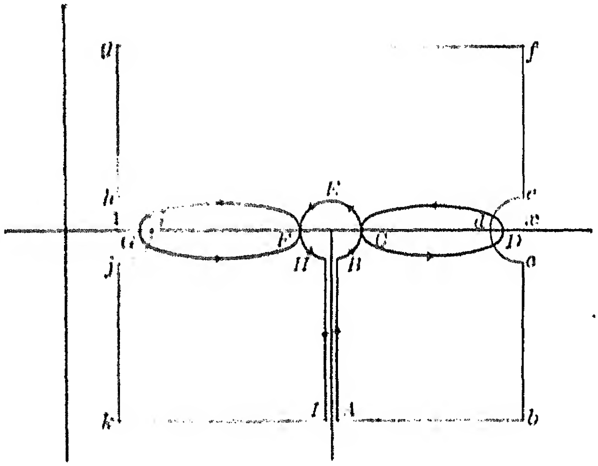

FIG. 2

respectively the points $w = 2, 3, 4, \cdots, p$ and $w = p + 1, p + 2, \cdots, x - 1$, where p is the integer for which $p < z < p + 1$; it being understood also that the closed curve $BCEFH$ forms a circle of arbitrarily small radius ξ with center at the point $w = z$.

It follows from (30) of Chapter I that we may then write

$$\sum_{r=2}^{x-1} \log (r - z) = \int_{CD} \frac{f_z(w)}{\varphi(w)}\, dw + \int_{GF} \frac{f_z(w)}{\varphi(w)}\, dw,$$

where CD and GF denote respectively that the indicated integrations take place in the positive sense along the closed contours CD and GF.

Whence, we have also

$$(24) \qquad \sum_{r=1}^{x-1} \log (r - z) = \log (1 - z) + \int_{C} \frac{f_z(w)}{\varphi(w)}\, dw - \int_{L} \frac{f_z(w)}{\varphi(w)}\, dw,$$

in which C indicates an integration over the entire contour from A to I, while L indicates an integration over the open loop $ABCEFHI$.

Let us now replace C, as may evidently be done, by the figure in part rectilinear and in part semicircular $AbcdefghijkI$ whose vertical sides (produced) pass respectively through the points $w = 1$, $w = x$. With the understanding that the radii of the arcs jih, cde are each equal to ϵ, we have now but to refer to the processes employed in § 8 to see that by taking $j = \infty$ we may write (24) in the form

$$\sum_{x=1}^{x-1} \log (x - z) = \tfrac{1}{2} \log (1 - z) - \Omega_{-z}(1) + \int_M \log (w - z) dw \tag{25}$$
$$- \tfrac{1}{2} \log (x - z) - \Omega_{-z}(x) + E(\epsilon) - \int_{L_\infty} \frac{f_z(w)}{\varphi(w)} dw,$$

where the path M extends from $w = 1$ to $w = x$ over the curve $1FEC x$, where L_∞ denotes the path resulting from L by placing $j = \infty$, where $\Omega_{-z}(x)$ is the expression obtained from (23) by replacing z by $-z$ and where $E(\epsilon)$ denotes an expression which becomes infinitesimal with ϵ and may therefore be at once neglected.

Now,

$$\int_M \log (w - z) dw = [(w - z) \log (w - z) - w]_{w=1}^{w=x} \tag{26}$$
$$= (z - 1) \log (1 - z) + 1 + (x - z) \log (x - z) - x.$$

Again, we may write

$$\int \frac{f_z(w)}{\varphi(w)} dw = \frac{1}{2\pi i} \log (w - z) \log (e^{-2\pi iw} - 1) - \frac{1}{2\pi i} \int \frac{\log (e^{-2\pi iw} - 1)}{w - z} dw$$

as appears by an integration once by parts. Whence,

$$\int_L \frac{f_z(w)}{\varphi(w)} dw = \log [e^{-2\pi i(z-ij)} - 1] - \log [e^{-2\pi iz} - 1].$$

Upon placing $j = \infty$ and making use of the relation $e^{-2\pi iz} - 1 = -2ie^{-\pi iz} \sin \pi z$ it thus appears that the last term of (25) (the coefficient -1 included) is equal to $-\log (-1) + \log (-2i) - \pi iz + \log \sin \pi z = \log 2i - \pi iz + \log \sin \pi z$.

Let us now combine relations (20), (21), (22), (25) and (26), availing ourselves also of (15) and of the facts just observed concerning the last term of (25). Noting mutual cancellation of terms and placing $x = \infty$ in the final result, we arrive at the relation

$$I(z) = \log \sin \pi z - \log 2\pi + \log 2i - \pi iz - \tfrac{1}{2} \log (1 - z^2)$$
$$+ z \log \frac{1 - z}{1 + z} + 2 - \Omega_z(1) - \Omega_{-z}(1).$$

If we now write

$$-\tfrac{1}{2} \log (1 - z^2) = - \log iz - \tfrac{1}{2} \log \left(1 - \frac{1}{z^2}\right); \qquad z \log \frac{1 - z}{1 + z} = \pi iz + z \log \frac{z - 1}{z + 1}$$

and then introduce the relation

$$-\log 2\pi - \log iz + \log 2i = -\log \pi z$$

we may therefore write

$$H(z) = \log \frac{\sin \pi z}{\pi z} - \tfrac{1}{2} \log \left(1 - \frac{1}{z^2}\right) + \left(2 + z \log \frac{z-1}{z+1}\right) - [\Omega_z(1) + \Omega_{-z}(1)].$$

Similarly, we arrive at the same result when z has a negative non-integral value.

Furthermore, the function

$$\Omega_z(1) + \Omega_{-z}(1) = -i \int_0^\infty \log \left[\frac{(1+iy)^2 - z^2}{(1-iy)^2 - z^2}\right] \frac{dy}{e^{2\pi y} - 1}$$

is single valued and analytic throughout the portions of the z plane lying to the right of the line $z = 1 + iy$ and to the left of the line $z = -1 + iy$ while the same function is developable asymptotically by Theorem V, Chapter I, throughout the same regions in the form

$$\frac{a_2}{z^2} + \frac{a_4}{z^4} + \frac{a_6}{z^6} + \cdots.$$

Noting that for the function $\Phi(z)$ defined in (19) we have $\Phi(-iz) = F(z)$ where $F(z)$ is the original function (12), and recalling also that

$$\frac{\sin \pi(-iz)}{\pi(-iz)} = \frac{e^{\pi z} - e^{-\pi z}}{2\pi z},$$

we may say in conclusion that *throughout any sector (vertex at $z = 0$) of the z plane which does not contain portions of the real axis, the function $F(z)$ defined by* (12) *may be developed asymptotically in the form*

$$F(z) \sim \frac{e^{\pi z} - e^{-\pi z}}{2\pi z}\left[1 + \frac{a_2}{z^2} + \frac{a_4}{z^4} + \cdots\right].$$

This result is seen to be consistent with (13).

24. We proceed to the following more general problem:

Example 3. Given

$$F(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{n^{1/\rho}}\right)$$

or

$$F(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{n^{1/\rho}}\right) \exp \sum_{\nu=1}^{p} \frac{1}{\nu}\left(\frac{z}{n^{1/\rho}}\right)^{\nu}; \qquad \begin{cases} p = \text{integer} \geqq 1 \\ p \leqq \rho < p+1 \end{cases}$$ [13]

according as $0 < \rho < 1$ or $\rho \geqq 1$. To determine asymptotic developments for $F(z)$.

[13] We adopt the familiar notation exp x for e^x.

This problem, in view of the important rôle which the $F(z)$ thus defined plays in the modern theory of integral functions, has already received considerable attention.[14] Our purpose here will be to deduce through a uniform method based on the fundamental theorems of Chapter I the known results together with others of a supplementary character.[15]

We shall suppose at first that ρ is non-integral and > 1 $(p < \rho < p + 1$, $p = integer \geqq 1)$. Also, for the present z is to be regarded as having any fixed value (real or complex) except one of the following: $2^{1/\rho}, 3^{1/\rho}, 4^{1/\rho}, \cdots$.

The method then requires that we take for consideration the expression (cf. (20))

$$(27)\qquad H(z) = \lim_{x=\infty}\left[\sum_{x=1}^{x-1}\log\,(x^\sigma - z) - \sigma\sum_{x=1}^{x-1}\log x + \sum_{x=1}^{x-1}\sum_{\nu=1}^{p}\frac{1}{\nu}\left(\frac{z}{x^\sigma}\right)^\nu\right]; \qquad \sigma = 1/\rho$$

in which the value to be assigned to $\log\,(x^\sigma - z)$ may for the present be taken in any manner consistent with the equation $\exp\log\,(x^\sigma - z) = x^\sigma - z$.

Then $\exp H(z) = F(z)$ where $F(z)$ is defined as above.

We proceed to study the behavior of the first term appearing in brackets in (27) when x is large.

Following the method of § 23, let us place

$$f_z(w) = \log\,(w^\sigma - z),$$

where w^σ is understood to be so defined as to be real when w is real and positive and where the logarithmic function is understood to be rendered single valued in w throughout the right half of the w-plane by means of a rectilinear cut extending from the point $w = z^\rho$ vertically downward to the point $w = \infty$, the value of z^ρ being determined in accordance with the following conventions: if $z = r(\cos\varphi + i\sin\varphi)$ then $z^\rho = r^\rho(\cos\rho\varphi + i\sin\rho\varphi)$ subject to the relation $-2\pi < \varphi \leqq 0$. The function $f_z(w)$ having been thus defined and defined uniquely for every value of w whose real part is positive, let us now impose for the present the additional condition upon z; viz., *real part of* $z^\rho > 1$; *i. e.*, $r^\rho\cos\rho\varphi > 1$. Next, let us consider the complex integral

$$\int_C \frac{f_z(w)}{\varphi(w)}\,dw; \qquad \varphi(w) = e^{2\pi i w} - 1$$

[14] See Mellin, *Acta Soc. Sc. Fennicae*, Vol. 29, No. 4 (1900); Barnes, *Philosophical Trans.*, Vol. 199A (1902); Lindelöf, *Acta Soc. Sc. Fennicae*, Vol. 31 (1902), p. 53; Wiman, *Arkiv för matem.*, Vol. 1 (1903), p. 105; Mattson, "Contributions à la théorie des fonctions entières" (Thèse, Upsala, 1905); Hardy, *Quarterly Journ. of Math.*, Vol. 37 (1906), pp. 146–172; Ford, *Annals of Math.*, Vol. 2 (2) (1910), pp. 115–127.

[15] It may be observed that this problem differs from the earlier more special ones of §§ 19, 20, 22 and 23 in that no formulae are at hand analogous to (2) and (13) by which we can predict beforehand the character of the solution. The present problem therefore illustrates to good advantage the value of the methods which we have been using.

taken from the point $w = z^\rho - ij$ (j = any real, positive value, arbitrarily large) situated on the right side of the above mentioned cut, around the open contour $C = ABCBDEFGHGI$ indicated in the following figure, which contour, as a result of the above condition $r^\rho \cos \rho\varphi > 1$, necessarily includes the point $w = z^\rho$ within its interior, it being here understood, as in Fig. 2, that the point I is the

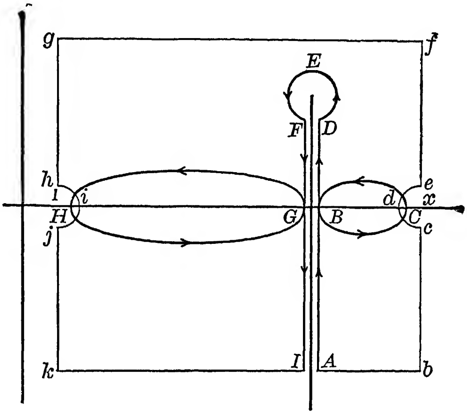

FIG. 3

one on the *left* side of the cut corresponding to A and that the closed loops BC and GH include respectively the points $w = 2, 3, 4, \cdots, q$ and $w = q + 1, q + 2, \cdots, (x - 1)$ where q is the integer for which $q <$ *real part* $z^\rho < q + 1$[16]; also that the curve DEF forms a circle of arbitrarily small radius ξ surrounding the point $w = z^\rho$.

Corresponding to relation (25) of § 23 we thus obtain

$$\sum_{x=1}^{x-1} \log (x^\sigma - z) = \tfrac{1}{2} \log (1 - z) - \Omega_z(1) + \int_M \log (w^\sigma - z) dw$$
$$- \tfrac{1}{2} \log (x^\sigma - z) + \Omega_z(x) - \int_L \frac{f_z(w)}{\varphi(w)} dw$$

where M indicates an integration over the path $1GFEDBx$,[17] where L indicates an integration over the path $ADEFI$ in which, however, the points A, I are now supposed to be taken at an infinite distance along the cut, and where $\Omega_z(x)$ is given by the formula

[16] In case *real part* $z^\rho = q =$ *an integer*, the indicated loop HG, instead of containing $w = q$ in its interior, will have this point upon its boundary. To obviate the difficulty thus arising, let it be understood in this case that the cut does not extend vertically downward from the point $w = z^\rho$ but first extends an arbitrarily small distance to the right of this point and then vertically downward as before. The reasoning which follows will then apply.

[17] In case z^ρ is real and > 1 the path M becomes the curve, in part rectilinear and in part semicircular, $1FECx$ of Fig. 2; while if *imag. part* $z^\rho < 0$ the path M may be taken as the straight line $1x$ (Fig. 3).

$$(28) \qquad \Omega_z(x) = -i\int_0^\infty \frac{\log\,[(x+iy)^\sigma - z] - \log\,[(x-iy)^\sigma - z]}{e^{2\pi y}-1}\,dy$$

or

$$(29) \qquad \Omega_z(x) = -i\int_0^\infty \log\left[\frac{(x+iy)^\sigma - z}{(x-iy)^\sigma - z}\right]\frac{dy}{e^{2\pi y}-1},$$

it being understood that the integrand of (29) is so defined as to be equal for all values of x and y to the integrand of (28).

Now, an integration once by parts shows that

$$\int \log\,(w^\sigma - z)dw = w\log\,(w^\sigma - z) - \sigma w - \sigma z\int\frac{dw}{w^\sigma - z}.$$

Whence,

$$\int_M \log\,(w^\sigma - z)dw = x\log\,(x^\sigma - z) - \sigma x - \sigma z\int_M \frac{dw}{w^\sigma - z} - \log\,(1-z) + \sigma.$$

We have now but to recall the formula (15) to see that the first two terms in the square bracket of (27) combine into the following:

$$(30) \qquad \begin{aligned}\sum_{x=1}^{x-1}\log\,(w^\sigma - z) - \sigma\sum_{x=1}^{x-1}\log x = {} & -\frac{\sigma}{2}\log 2\pi - \tfrac{1}{2}\log\,(1-z)\\ & - \Omega_z(1) + \omega(x) + \sigma + (x-\tfrac{1}{2})\log\left(1-\frac{z}{x^\sigma}\right)\\ & - \sigma z\int_M \frac{dw}{w^\sigma - z} + \Omega_z(x) - \int_L \frac{f_z(w)}{\varphi(w)}\,dw.\end{aligned}$$

We turn next to consider the third term appearing in square brackets in (27). By use of the well-known relation[18]

$$\zeta(t) = 1 + \frac{1}{2^t} + \frac{1}{3^t} + \cdots + \frac{1}{(n-1)^t} + \frac{n^{1-t}}{t-1} + \theta_t(n); \qquad \begin{cases} \textit{real part } p > 0,\\ \lim_{n=\infty}\theta_t(n) = 0\end{cases}$$

wherein ζ represents the Riemann ζ function, we may write

$$(31) \qquad \begin{aligned}\sum_{x=1}^{x-1}\sum_{\nu=1}^{p}\frac{1}{\nu}\left(\frac{z}{x^\sigma}\right)^\nu &= \sum_{\nu=1}^{p}\zeta(\sigma\nu)\frac{z^\nu}{\nu} + \sum_{\nu=1}^{p}\frac{z^\nu}{\nu}\left(\frac{x^{1-\sigma\nu}}{1-\sigma\nu}\right) + \xi_z(x)\\ &= \sum_{\nu=1}^{p}\zeta(\sigma\nu)\frac{z^\nu}{\nu} + \sum_{\nu=1}^{p}\frac{z^\nu}{\nu x^{\sigma\nu-1}} + \sigma\sum_{\nu=1}^{p}\frac{z^\nu}{(1-\sigma\nu)x^{\sigma\nu-1}} + \xi_z(x);\\ &\qquad\qquad \lim_{x=\infty}\xi_z(x) = 0.\end{aligned}$$

Whence, upon observing that the sixth term in the second member of (30) is of the form

$$-\sum_{\nu=1}^{\infty}\frac{z^\nu}{\nu x^{\sigma\nu-1}} + \eta(x); \qquad \lim_{x=\infty}\eta(x) = 0$$

[18] See PETERSEN, "Vorlesungen über Funktionstheorie" (Copenhagen, 1898), pp. 161–169.

also that $\lim_{x \to \infty} \Omega_s(x) = 0$, we arrive at the following relation after combining (27), (30) and (31) and placing $x = \infty$

$$(32) \quad H(z) = -\frac{\sigma}{2}\log 2\pi - \tfrac{1}{2}\log(1-z) - \Omega_s(1) + \sum_{\nu=1}^{p} \zeta(\sigma\nu)\frac{z^\nu}{\nu} + S(z) - \int_L \frac{f_s(w)}{\varphi(w)}\,dw,$$

where

$$(33) \qquad S(z) = \lim_{x=\infty} \sigma\left[1 + \sum_{\nu=1}^{p}\frac{z^\nu}{(1-\sigma\nu)x^{\sigma\nu-1}} - z\int_M \frac{dw}{w^\sigma - z}\right].$$

The properties of $S(z)$ will be considered in further detail later. For the present we turn to the last term of (32).

By placing

$$u = f_s(w) = \log(w^\sigma - z), \qquad dv = \frac{dw}{\varphi(w)} = \frac{dw}{e^{2\pi i w} - 1}$$

so that

$$du = \frac{\sigma w^{\sigma-1}}{w^\sigma - z}dw, \qquad v = \frac{1}{2\pi i}\log(e^{-2\pi i w} - 1)$$

it appears that

$$(34) \quad \int \frac{f_s(w)}{\varphi(w)}dw = \frac{1}{2\pi i}\log(w^\sigma - z)\log(e^{-2\pi i w} - 1) - \frac{\sigma}{2\pi i}\int \frac{w^{\sigma-1}\log(e^{-2\pi i w} - 1)}{w^\sigma - z}dw.$$

Now, the difference between the value of the first term here appearing on the right when considered at the point $w = A$ and its value at the point $w = I$ is

$$(35) \qquad \log[-1 + \exp - 2\pi i\,(\textit{real part } z^\rho - ij)]$$

as appears by making the substitution $w^\sigma = s$ or $w = s^\rho$ and evaluating the resulting expression between corresponding s limits. Moreover, by means of the same substitution the second term in the right member of (34) becomes

$$(36) \qquad -\frac{1}{2\pi i}\int \frac{\log[-1 + \exp - 2\pi i s^\rho]}{s - z}ds,$$

where the integration is extended over a contour in the s plane which includes no singularities of the integrand except the simple pole at $s = z$. But the value of (36) is evidently the negative of the residue of the integrand at the point $s = z$; *i. e.*, $-\log[-1 + \exp - 2\pi i z^\rho]$.

Since the expression (35) becomes $\log(-1)$ in the limit as $j = \infty$, it thus appears that

$$(37) \qquad \int_L \frac{f_s(w)}{\varphi(w)}dw = \log(-1) - \log[-1 + \exp - 2\pi i z^\rho].$$

If in (32) we place $\log(1-z) = \log(-z) + \log(1 - 1/z)$ we may there-

fore write the original function $F(z)$ in the form

(38)
$$F(z) = A(z)B(z),$$

where

$$A(z) = \frac{e^{-2\pi i z^\rho} - 1}{-\sqrt{-z}\sqrt[2\rho]{2\pi}} = \frac{e^{-2\pi i z^\rho} - 1}{-i\sqrt[2\rho]{2\pi z^\rho}} = \frac{2e^{-\pi i z^\rho}\sin \pi z^\rho}{\sqrt[2\rho]{2\pi z^\rho}}$$

and

(39)
$$B(z) = \exp\left[\sum_{\nu=1}^{p} \zeta(\sigma\nu)\frac{z^\nu}{\nu} + S(z) - \Omega_z(1) - \tfrac{1}{2}\log\left(1 - \frac{1}{z}\right)\right].$$

Thus far we have supposed z to have any value (real or complex) such that $z \neq n^{1/\rho}$; $n = 1, 2, 3, \cdots$, and such that having placed $z = r(\cos\varphi + i\sin\varphi)$ and agreed to write $z^\rho = r^\rho(\cos\rho\varphi + i\sin\rho\varphi)$ with $-2\pi < \varphi \leqq 0$, we have $r^\rho\cos\rho\varphi > 1$. We now proceed to study in further detail the expression $B(z)$ and for this it is desirable to remove for the present all restrictions as regards z, thus enabling us to determine certain functional properties of the same expression.

We turn first to the expression $\Omega_z(1)$ which appears in $B(z)$ and which by reference to (29) is seen to be defined by the relation

(40)
$$\Omega_z(1) = -i\int_0^\infty \log\left[\frac{(1+iy)^\sigma - z}{(1-iy)^\sigma - z}\right]\frac{dy}{e^{2\pi y} - 1}.$$

For a given value, real or complex, of z this $\Omega_z(1)$ evidently has a meaning unless z be such that the equation $(1 \pm iy)^\sigma = z$ has a *real* solution in y. In order to determine the values of z for which this happens, let us place

$$z = r(\cos\varphi + i\sin\varphi)$$

and make the conventions already indicated as to the meaning of z^σ. For the exceptional values in question we must then have $1 \pm iy = r^\rho(\cos\rho\varphi + i\sin\rho\varphi)$ so that the same values are those lying on the locus of the equation $r^\rho\cos\rho\varphi = 1$; $-2\pi < \varphi \leqq 0$. Whence, if r be large the same values will tend to have an argument of the form $-(2n+1)\pi/2\rho$ wherein n is a positive integer for which the same argument lies between -2π and 0. Again, if the locus just mentioned be drawn, the z plane is thereby divided into portions in each of which $\Omega_z(1)$ is a single valued, analytic function of z, since within any sub-region T' lying wholly within such portion the convergence of the integral in (40) is readily seen to be *uniform*. Moreover, if *arg* z has any value other than one of the exceptional type just mentioned, we may write $\lim\limits_{|z|=\infty} \Omega_z(1) = 0$. In fact, upon reference to Theorem V, Chapter I, it appears that under such hypotheses we shall have

$$\Omega_z(1) \sim \frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \cdots,$$

where the coefficients $a_1, a_2, \cdots$ may be obtained by expanding the integrand of (40) in ascending powers of $1/z$ and integrating term by term.

Secondly, we turn to the expression $S(z)$ defined by (33). Since

$$\frac{z}{w^\sigma - z} = \sum_{\nu=1}^{p} \frac{z^\nu}{w^{\sigma\nu}} + \frac{z^{p+1}}{w^{\sigma p}(w^\sigma - z)},$$

we may write

$$z\int \frac{dw}{w^\sigma - z} = \sum_{\nu=1}^{p} \frac{z^\nu}{(1-\sigma\nu)w^{\sigma\nu-1}} + z^{p+1}\int \frac{dw}{w^{\sigma p}(w^\sigma - z)}. \tag{41}$$

Whence, recalling that M extends from $w = 1$ to $w = x$, we obtain the following relation:

$$S(z) = \sigma\left[\sum_{\nu=0}^{p} \frac{z^\nu}{1-\sigma\nu} - z^{p+1}\int_N \frac{dw}{w^{\sigma p}(w^\sigma - z)}\right], \tag{42}$$

where N represents the path obtained from M by supposing $x = +\infty$.

In the consideration of this expression we have thus far considered that *real part* $z^\rho > 1$ and from what has already been noted it follows that if we have also *imag part* $z^\rho < 0$ we may replace (42) by

$$S(z) = \sigma\left[\sum_{\nu=0}^{p} \frac{z^\nu}{1-\sigma\nu} - z^{p+1}\int_1^\infty \frac{dx}{x^{\sigma p}(x^\sigma - z)}\right]. \tag{43}$$

The form of (42) may also be simplified when *imag part* $z^\rho > 0$ (*real part* $z^\rho > 1$). In fact, we may then write

$$\int_N \frac{dw}{w^{\sigma p}(w^\sigma - z)} = \int_1^\infty \frac{dx}{x^{\sigma p}(x^\sigma - z)} - \int_{c_\rho} \frac{dw}{w^{\sigma p}(w^\sigma - z)}, \tag{44}$$

where the last symbol represents an integration in the positive sense about the circle.

Moreover, the last term of (44) is readily evaluated and found to be equal to $2\pi i\rho z^{\rho-p-1}$.

Thus, when *imag part* $z^\rho > 0$ (*real part* > 1) we may write

$$S(z) = \sigma\left[\sum_{\nu=0}^{p} \frac{z^\nu}{1-\sigma\nu} - z^{p+1}\int_1^\infty \frac{dx}{x^{\sigma p}(x^\sigma - z)}\right] + 2\pi i\rho z^\rho. \tag{45}$$

These facts being premised, let us consider the properties of the right member of (43), all assumptions as regards z being laid aside for the moment. Evidently, the expression in question represents a function of z which is single valued and analytic in any region T which does not cut the portion of the real z axis extending from $z = 1$ to $z = +\infty$. Moreover, when $|z| < 1$ we find upon expanding in ascending powers of z that the same expression is developable in the form

$$\sigma\sum_{\nu=0}^{\infty} \frac{z^\nu}{1-\sigma\nu} = \sum_{\nu=0}^{\infty} \frac{z^\nu}{\rho - \nu}. \tag{46}$$

For *large* values of $|z|$ the properties of the right member of (43) are now

derivable by means of the following Lemma which we shall state and this point and to which we shall have occasion to refer frequently th what follows.

Lemma.[19] If the coefficient $g(n)$ of the power series

$$(47)\qquad \sum_{n=a}^{\infty} g(n)z^n; \qquad \begin{array}{l} a = \textit{integer, positive, negative or zero} \\ r = \textit{radius of convergence} > 0 \end{array}$$

is such that (*a*) when considered as a function $g(w)$ of the complex $w = x + iy$ it is single valued and analytic throughout all portions plane lying to the right of (or upon) the vertical line $w = a - \frac{1}{2} + iy$ e a finite number of poles situated at the points $w = \lambda_1, \lambda_2, \cdots, \lambda_t, \cdots, \lambda_m$; *teger* $\geqq a$[20] and (*b*) is such that to an arbitrarily small positive quantit corresponds a positive constant K (independent of x and y) such that

$$\left|\frac{g(x \pm iy)}{g(x)}\right| < K \exp \epsilon y \qquad {}^{21}$$

for all values of $x \geqq a - \frac{1}{2}$ and for all positive values of y sufficiently la the function $f(z)$ defined by (47) when $|z| < r$ may be extended an throughout the whole z plane with the exception of the positive ha real axis, and throughout this region will be defined by the equation

$$(48)\qquad f(z) = \frac{(-1)^a}{2}\int_{-\infty}^{\infty} \frac{g(a - \frac{1}{2} + iy)(-z)^{a-\frac{1}{2}-iy}}{\cosh \pi y}\, dy - \sum_{t=1}^{m} r_t$$

in which, if we place $z = r(\cos\varphi + i \sin\varphi)$ it is supposed that we writ

$$(-z)^{a-\frac{1}{2}-iy} = \exp\left[(a - \tfrac{1}{2} + iy)\log(-z)\right]$$

$$= \exp\left[(a - \tfrac{1}{2} + iy)(\log r + i\right.$$

and take $-2\pi < \varphi < 0$ and in which r_t represents the residue of the fu

$$(49)\qquad \frac{\pi g(w)(-z)^w}{\sin \pi w}$$

at the pole $w = \lambda_t$."

For the proof of this lemma let us at first suppose for simplicity t has no poles at the right of (or upon) the vertical line $w = a - \frac{1}{2} + i$ us regard z for the present as having any fixed value. The lemma the from a consideration of the result obtained by integrating the funct

[19] Cf. LINDELÖF, "Calcul des résidus" (Paris, 1905), p. 109; also, FORD, *Journ.* Vol. 9 (5) (1903), p. 223; also, *Bulletin of Amer. Math. Soc.*, Vol. 16 (2) (1910), p. 507.

[20] This condition is fulfilled from the fact that $g(n)$ has a meaning when $n = a, a +$ $\cdots$. Otherwise the given series (47) would lose significance.

[21] This condition is satisfied in particular if constants $C_1 > 0$ and $C_2 \geqq C_1$ exist $C_1 < |g(w)| < C_2$.

about the rectangular contour C_n formed in the w plane by the lines

$$w = a - \tfrac{1}{2} + iy, \qquad w = \tfrac{1}{2} + 2n + iy, \qquad w = x \pm ij$$

where n is any integer such that $2n > a$ and where j is any positive quantity arbitrarily large. Upon applying (30) of Chapter I to the result of such an integration, we arrive in the first place at the relation

$$\sum_{n=a}^{2n} g(n)z^n = \frac{1}{2i}\int_{C_n} \frac{g(w)(-z)^w}{\sin \pi w}\,dw. \tag{50}$$

Supposing at first that z is real and negative, we proceed to study the integral here appearing in further detail.

First, along the side of C_n upon which $w = x + ij$ we have $dw = dx$ and $\sin \pi w = \sin \pi(x + ij) = \sinh \pi j(\sin \pi x \coth \pi j + i \cos \pi x)$ so that if we call the contribution from the side in question I, we may write

$$I = \frac{(-z)^{ij}}{2i \sinh \pi j}\int_{2n+\frac{1}{2}}^{a-\frac{1}{2}} \frac{g(x+ij)(-z)^x}{\sin \pi x \coth \pi j + i \cos \pi x}\,dx.$$

Whence $\lim\limits_{j=\infty} I = 0$ provided that

$$\lim_{j=\infty} e^{-\pi j} g(x + ij) = 0; \qquad x \geqq a - \tfrac{1}{2}. \tag{51}$$

Similarly, we find the same result for the contribution arising from the side of C_n upon which $w = x - ij$ provided, however, that

$$\lim_{j=\infty} e^{-\pi j} g(x - ij) = 0; \qquad x \geqq a - \tfrac{1}{2}. \tag{52}$$

We observe that both conditions (51) and (52) are satisfied in the present case as a result of (b) of our hypotheses.

Next, let us consider the side of C_n upon which $w = \frac{1}{2} + 2n + iy$. Here we have $dw = idy$, $\sin \pi w = \cos i\pi y = \cosh \pi y$ so that having taken $j = \infty$, the contribution in question becomes

$$J = \frac{(-z)^{\frac{1}{2}+2n}}{2}\int_{-\infty}^{\infty} \frac{g(\frac{1}{2} + 2n + iy)(-z)^{iy}}{\cosh \pi y}\,dy,$$

and it follows from (b) of our hypotheses that the improper integral here appearing has a meaning (z real and negative). Moreover, it follows likewise that if $|z| < r$ we shall have $\lim\limits_{n=\infty} J = 0$.

Whence, if we now take account of the contribution arising from the remaining side $w = a - \frac{1}{2} + iy$ of C_n, noting that we here have $\sin \pi w = (-1)^{a-1} \cosh \pi y$ while the integration takes place from $y = +\infty$ to $y = -\infty$, we may write

$$\sum_{n=a}^{\infty} g(n)z^n = \frac{(-1)^a}{2}\int_{-\infty}^{\infty} \frac{g(a - \frac{1}{2} + iy)(-z)^{a-\frac{1}{2}+iy}}{\cosh \pi y}\,dy. \tag{53}$$

This relation must hold good as we have indicated, for all values of z which are real and negative and such that $|z| < r$. But the first member represents a function of the complex variable z which is single valued and analytic throughout the circle of convergence of (47) while the second member, with the conventions introduced in the lemma as regards the meaning of $(-z)^{a-\frac{1}{2}+iy}$ represents a function of z which is single valued and analytic throughout the whole z plane except for the positive half of the real axis. In fact, for all values of z in a region T which does not cut or touch the positive half of the real axis we shall have from the indicated conventions $-\pi < \varphi < \pi$ so that upon introducing (b) of the hypotheses it appears that we may choose ϵ so small that the improper integral in (53) will converge *uniformly* for all values of z in T. Whence, the same integral will have the analytic properties just indicated, and we reach in summary the lemma for the case in which $g(w)$ has no poles to the right of the line $w = a - \frac{1}{2} + iy$.

That the lemma holds true in the more general case follows at once upon noting that relation (50) then continues (n sufficiently large) provided we add to its first member the expression

$$\sum_{t=1}^{m} r_t.$$

Returning to the second member of (43) which is defined when $|z| < 1$ by the series (46), let us now apply the above lemma to the latter series, taking for this purpose $g(w) = 1/(\rho - w)$. Since the residue of the function

$$\frac{\pi(-z)^w}{(\rho - w)\sin \pi\rho} \tag{54}$$

at the pole $w = \rho$ is $-\pi(-z)^\rho/\sin \pi\rho$, it thus appears that for all values of z except those real and positive we may write the expression in question in the form

$$\frac{\pi(-z)^\rho}{\sin \pi\rho} + R(z),$$

where

$$R(z) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{(-z)^{-\frac{1}{2}+iy}}{(\rho + \frac{1}{2} - iy)\cosh \pi y}\,dy,$$

it being here understood that the expressions $(-z)^\rho$ and $(-z)^{-\frac{1}{2}+iy}$ are to be interpreted in accordance with the conventions stated in the lemma, *i. e.*, if $z = r(\cos\varphi + i\sin\varphi)$ with $-2\pi < \varphi \leqq 0$, then

$$(-z)^\rho = \exp \rho(\log r + i\varphi + i\pi) = r^\rho[\cos \rho(\varphi + \pi) + i \sin \rho(\varphi + \pi)]$$

and

$$(-z)^{-\frac{1}{2}+iy} = \exp\,[(-\tfrac{1}{2} + iy)(\log r + i\varphi + i\pi)].$$

Upon referring to (43) and (44) it follows then, as regards the expression

$S(z)$ originally defined by (33), that throughout any region T_1 of the z plane in which *real part* $z^\rho > 1$, *imag part* $z^\rho < 0$ we shall have

(55) $$S(z) = \frac{\pi(-z)^\rho}{\sin \pi\rho} + R(z),$$

while throughout any similar region T_2 in which *real part* $z^\rho > 1$, *imag part* $z^\rho > 0$ we shall have

$$S(z) = \frac{\pi(-z)^\rho}{\sin \pi\rho} + 2\pi i\rho z^\rho + R(z).$$

Hence, according as z lies in T_1 or T_2 the expression $B(z)$ defined by (39) takes the form

$$B(z) = \exp C(z) \quad \text{or} \quad B(z) = \exp\,[C(z) + 2\pi i\rho z^\rho],$$

where

$$C(z) = \sum_{\nu=1}^{p} \zeta\left(\frac{\nu}{\rho}\right)\frac{z^\nu}{\nu} + \frac{\pi(-z)^\rho}{\sin \pi\rho} - \tfrac{1}{2}\log\left(1 - \frac{1}{z}\right) - \Omega_z(1) + R(z).$$

We note also that since $R(z)$ is equal to $1/2\pi i$ multiplied by the result of integrating the expression (54) from $y = -\infty$ to $y = +\infty$ along the line $w = -\frac{1}{2} + iy$ it follows that we may replace $R(z)$ by a similar expression $\bar{R}(z)$ in which the path of integration is $w = -k - \frac{1}{2} + iy$ (k = arbitrarily large positive integer) provided this $\bar{R}(z)$ be increased by the sum of the residues of the function (54) at the poles $w = -1, -2, \cdots, -k$. Moreover, since these residues form the first k terms of a series of the form

(56) $$a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots; \qquad a_1, a_2, \cdots \textit{ constants as regards } z$$

while $\lim_{|z|=\infty} z^k\bar{R}(z) = 0$ it follows that the original expression $R(z)$ is developable asymptotically in the form (56) ($arg\ z \neq 0$).

It follows, then, upon reference to (38) and to the properties which we have now established for $\Omega_z(1)$ and $R(z)$ that we shall have the following relation in which the upper or lower of the double sign $\pm$ is to be taken according as z is confined to T_2 or T_1:

(57) $$F(z) \sim \frac{2\sin \pi z^\rho}{\sqrt[2\rho]{2\pi z^\rho}} \exp\left[\pm\, \pi i z^\rho + \sum_{\nu=1}^{p}\zeta\left(\frac{\nu}{\rho}\right)\frac{z^\nu}{\nu} + \pi\frac{(-z)^\rho}{\sin \pi\rho}\right].$$

Upon observing that when *imag part* $z^\rho > 0$ the function $\exp \pi i z^\rho$ is developable asymptotically in the form (56) with $a_0 = 0$, $a_1 = 0$, $\cdots$, while the same is true of the function $\exp - \pi i z^\rho$ when *imag part* $z^\rho < 0$, it appears that the above relation may be simplified into the following holding good for values of z in regions of either type T_1 or T_2:

(58) $$F(z) \sim \frac{2\sin \pi z^\rho}{\sqrt[2\rho]{2\pi z^\rho}} \exp\left[\sum_{\nu=1}^{p}\zeta\left(\frac{\nu}{\rho}\right)\frac{z^\nu}{\nu} + \pi\frac{(-z)^\rho}{\sin \pi\rho}\right].$$

This relation, as we have noted, holds true only when *real part* $z^\rho > 1$. We now proceed to determine an analogous relation for any region T_3 in which *real part* $z^\rho < 1$.

If this assumption be made at the beginning, the cut in the w plane falls entirely outside the rectangle *bfgk* so that we at once obtain (32) except that the last term of the second member is lacking. Moreover, the expression $S(z)$ takes the form (55) so that, upon writing $\log(1-z) = \log(-z) + \log(1-(1/z))$, we have

$$H(z) = -\frac{1}{2\rho}\log 2\pi - \frac{1}{2\rho}\log(-z)^\rho - \tfrac{1}{2}\log\left(1-\frac{1}{z}\right) - \Omega_s(1) + \sum_{\nu=1}^{p}\zeta\left(\frac{\nu}{\rho}\right)\frac{z^\nu}{\nu} + \frac{\pi(-z)^\rho}{\sin\pi\rho} + R(z) \tag{59}$$

and hence

$$F(z) \sim \frac{1}{\sqrt[2\rho]{2\pi(-z)^\rho}}\exp\left[\sum_{\nu=1}^{p}\zeta\left(\frac{\nu}{\rho}\right)\frac{z^\nu}{\nu} + \frac{\pi(-z)^\rho}{\sin\pi\rho}\right]. \tag{60}$$

Before summarizing the preceding results into a theorem it is desirable to note certain corresponding results which may be obtained when z^ρ is confined to the real domain $(1, +\infty)$—a case not included in the above discussion.

If this assumption be made at the beginning the corresponding Fig. 3 becomes that represented in Fig. 2, except that the cut extends from the point $w = z^\rho$ instead of from the point $w = z$. Thus we obtain equation (32) as before with $S(z)$ defined by (33) in which, however, the path M is now understood to be $1FECx$ of Fig. 2. We arrive, therefore, at (38) in which $A(z)$ is defined as before, while $B(z)$ is defined by (39) with $S(z)$ given by (42) wherein N represents the path $1FEC+\infty$. In this form $S(z)$ is now developable (as was (43)) when $|z| < 1$ into the form (46) from which we find as before that unless

$$\varphi = \arg z = 0$$

the expression $S(z)$ is given by (55). In other words, $S(z)$ will be given by (55) when φ has any one of the following values (for which z^ρ as above defined is real and positive) except the value $\varphi = 0$:

$$0,\ -\frac{2\pi}{\rho},\ -\frac{4\pi}{\rho},\ -\frac{6\pi}{\rho},\ \cdots,\ -\frac{2k\pi}{\rho};\qquad -\frac{2k\pi}{\rho} > -2\pi. \tag{61}$$

Moreover, when $\varphi = 0$, $S(z)$ preserves a meaning as appears from (42), provided that $z \neq 1$, so that the same expression may be obtained from (55) when $z = r$ is large by placing therein $z = r(\cos\varphi + i\sin\varphi)$, observing the indicated conventions as to the meaning of $(-z)^\rho$, $(-z)^{-\frac{1}{2}+i\nu}$ and passing to the limit in the resulting expression as φ approaches the value zero through *negative* values.

Thus it appears that when z^ρ is real and positive—*i. e.*, when φ has any

one of the values (61) —we shall have relation (57) in which the negative sign is to be taken before the expression $\pi i z^\rho$ which appears in the square bracket.

Upon noting the various sections of the z plane which correspond respectively to regions of the types T_1, T_2 and T_3, we thus arrive at the following

THEOREM I. *Given the typical integral function of rank p (order > 0):*

$$F(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{n^{1/\rho}}\right)\exp\sum_{\nu=1}^{p}\frac{1}{\nu}\left(\frac{z}{n^{1/\rho}}\right)^{\nu}; \qquad p \doteq \textit{integer} \geqq 1$$

with the assumption that ρ is such that $p < \rho < p + 1$.

If, having placed $z = r(\cos\varphi + i\sin\varphi)$ we agree that z^ρ and $(-z)^\rho$ shall be defined respectively by the equations

$$z^\rho = r^\rho(\cos\rho\varphi + i\sin\rho\varphi); \qquad -2\pi < \varphi \leqq 0$$

$$(-z)^\rho = r^\rho[\cos\rho(\varphi + \pi) + i\sin\rho(\varphi + \pi)]$$

then for values of z of large modulus and lying within sectors of the type

$$-\frac{4k+3}{2\rho}\pi < \varphi < -\frac{4k+1}{2\rho}\pi; \qquad k = 0, 1, 2, 3, \cdots; \qquad -2\pi < \varphi < 0$$

we shall have

$$F(z) \sim \frac{1}{\sqrt[2\rho]{2\pi(-z)^\rho}}\exp\left[\sum_{\nu=1}^{p}\zeta\left(\frac{\nu}{\rho}\right)\frac{z^\nu}{\nu} + \pi\frac{(-z)^\rho}{\sin\pi\rho}\right],$$

where ζ is the symbol for the Riemann ζ function, while for values of z of large modulus and lying within sectors of the type

$$-\frac{4k+1}{2\rho}\pi < \varphi < -\frac{4k-1}{2\rho}\pi; \qquad k = 0, 1, 2, 3, \cdots; \qquad -2\pi < \varphi \leqq 0$$

we shall have

$$F(z) \sim \frac{2\sin\pi z^\rho}{\sqrt[2\rho]{2\pi z^\rho}}\exp\left[\sum_{\nu=1}^{p}\zeta\left(\frac{\nu}{\rho}\right)\frac{z^\nu}{\nu} + \pi\frac{(-z)^\rho}{\sin\pi\rho}\right],$$

provided φ does not have one of the exceptional values $0, -2\pi/\rho, -4\pi/\rho, -6\pi/\rho, \cdots$.

Moreover, for the exceptional values of φ just mentioned we shall have when $|z|$ is large

$$F(z) \sim \frac{2\sin\pi z^\rho}{\sqrt[2\rho]{2\pi z^\rho}}\exp\left[-\pi i z^\rho + \sum_{\nu=1}^{p}\zeta\left(\frac{\nu}{\rho}\right)\frac{z^\nu}{\nu} + \pi\frac{(-z)^\rho}{\sin\pi\rho}\right].$$

In the following figure the sectorial regions indicated, I and II, represent those in which for large values of $|z|$ the first or second of the above forms holds good respectively, while the dotted lines represent the special directions along which the third form applies. It is to be understood that the last radial

line drawn is that upon which $\varphi = -9\pi/2\rho$, but that a complete figure would contain all similar lines upon which

$$\varphi = -\frac{2k+1}{2\rho}\pi; \qquad k = 0, 1, 2, \cdots \quad \text{and} \quad \varphi > -2\pi,$$

the scheme of alternate division of the plane into sectors of types I and II being carried forward up to and including the last sector thus obtained.

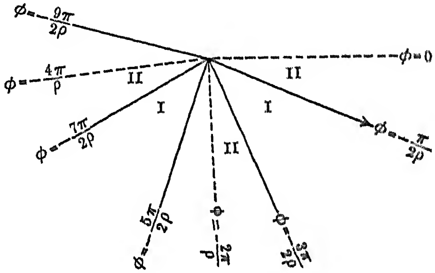

FIG. 4

Upon noting that for values of z which are real and positive ($z = r$) we have

$$-\pi i z^\rho + \frac{\pi(-z)^\rho}{\sin \pi\rho} = -\pi i r^\rho + \pi r^\rho \frac{\cos \rho\pi + i \sin \rho\pi}{\sin \rho\pi} = \pi r^\rho \cot \rho\pi,$$

it appears that the above theorem is consistent with certain results of HARDY to be found in the *Quarterly Journal of Mathematics*, Vol. 37 (1905), page 158 (later corrected on page 373). For values of z for which $arg\ z = \varphi \neq 0$ the theorem is not altogether consistent with the results of BARNES in the *Philosophical Transactions*, Vol. 199*A* (1902), page 470, since an equivalent to the first of the forms above is there assigned to $F(z)$ for *all* values of z such that $\varphi \neq 0$ ($|z|$ sufficiently large). It is to be observed that both BARNES and HARDY take for discussion the function $F(-z)$ instead of the $F(z)$ employed above.

25. In the discussion of the function $F(z)$ of § 24 we have thus far supposed $p < \rho < p+1$ where p is any integer $\geqq 1$. The corresponding results for cases in which $0 < \rho < 1$ may now be readily supplied, it being understood that $F(z)$ assumes the first of the forms given at the beginning of § 24.

Proceeding as in § 24, we obtain equation (27) as before except that the third term in square brackets is lacking. Whence, equation (32) continues except that the terms involving the function ζ are absent, while instead of (33) we have

$$S(z) = \lim_{x=\infty} \sigma \left[1 - z \int_M \frac{dw}{w^\sigma - z} \right].$$

Thus it appears at once that *the theorem of § 24 holds true when* $0 < \rho < 1$

provided that the term

$$\sum_{\nu=1}^{p} \zeta\left(\frac{\nu}{\rho}\right)\frac{z^\nu}{\nu}$$

there appearing be then omitted.

26. We proceed to consider the remaining cases—viz., those in which $\rho = p =$ *an integer*. The function $F(z)$ is then defined by the second of the forms appearing at the beginning of § 24.

Equations (27) and (30) are now obtained as before, but instead of (31) we write

$$\sum_{x=1}^{x-1}\sum_{\nu=1}^{p}\frac{1}{\nu}\left(\frac{z}{x^\sigma}\right)^\nu = \sum_{x=1}^{x-1}\sum_{\nu=1}^{p-1}\frac{1}{\nu}\left(\frac{z}{x^\sigma}\right)^\nu + \sum_{x=1}^{x-1}\frac{1}{p}\left(\frac{z}{x^\sigma}\right)^p$$

$$= \sum_{\nu=1}^{p-1}\zeta(\sigma\nu)\frac{z^\nu}{\nu} + \sum_{\nu=1}^{p-1}\frac{z^\nu}{\nu x^{\sigma\nu-1}} + \sum_{\nu=1}^{p-1}\frac{z^\nu}{\nu(1-\sigma\nu)x^{\sigma\nu-1}} + \sigma z^p \sum_{x=1}^{x-1}\frac{1}{x} + \theta_1(x).$$

Moreover, the last sum here appearing is evidently of the form

$$c + \log x + \theta_2(x); \qquad \lim_{x=\infty}\theta_2(x) = 0$$

where c represents Euler's constant.

Instead of (32) we thus obtain in the present case

$$H(z) = -\frac{\sigma}{2}\log 2\pi - \tfrac{1}{2}\log(1-z) - \Omega_z(1) + \sum_{\nu=1}^{p-1}\zeta(\sigma\nu)\frac{z^\nu}{\nu} \tag{62}$$
$$+ c\sigma z^p + S(z) - \int_L \frac{f_z(w)}{\varphi(w)}\,dw,$$

where

$$S(z) = \lim_{x=\infty}\left[\sigma + (x-\tfrac{1}{2})\log\left(1-\frac{z}{x^\sigma}\right) + \sigma z^p\log x \right. \tag{63}$$
$$\left. + \sum_{\nu=1}^{p-1}\frac{z^\nu}{\nu(1-\sigma\nu)x^{\sigma\nu-1}} - \sigma z\int_M\frac{dw}{w^\sigma - z}\right].$$

The last term of (62) may now be evaluated as before, leading to equations (37) and (38) in the latter of which $A(z)$ is defined as before while $B(z)$ is now defined by the relation

$$B(z) = \exp\left[\sum_{\nu=1}^{p-1}\zeta(\sigma\nu)\frac{z^\nu}{\nu} + c\sigma z^p + S(z) - \Omega_z(1) - \tfrac{1}{2}\log\left(1-\frac{1}{z}\right)\right]. \tag{64}$$

In order to study the functional properties of the present function $S(z)$ we first note that

$$(x-\tfrac{1}{2})\log\left(1-\frac{z}{x^\sigma}\right) = -\sum_{\nu=1}^{\infty}\frac{z^\nu}{\nu x^{\sigma\nu-1}} + \theta_3(x); \qquad \lim_{x=\infty}\theta_3(x) = 0,$$

$$\sum_{\nu=1}^{p-1}\frac{z^\nu}{\nu(1-\sigma\nu)x^{\sigma\nu-1}} = \sum_{\nu=1}^{p-1}\frac{z^\nu}{\nu x^{\sigma\nu-1}} + \sigma\sum_{\nu=1}^{p-1}\frac{z^\nu}{(1-\sigma\nu)x^{\sigma\nu-1}};$$

also that instead of (41) we may now write

$$z\int\frac{dw}{w^\sigma - z} = \sum_{\nu=1}^{p-1}\frac{z^\nu}{(1-\sigma\nu)w^{\sigma\nu-1}} + z^p \log w + z^{p+1}\int\frac{dw}{w(w^\sigma - z)}.$$

Whence,

$$(65)\quad \begin{aligned} S(z) &= \lim_{x=\infty}\left[\sigma - \sum_{\nu=p}^{\infty}\frac{z^\nu}{\nu x^{\sigma\nu-1}} + \sigma\sum_{\nu=1}^{p-1}\frac{z^\nu}{1-\sigma\nu} - \sigma z^{p+1}\int_M\frac{dw}{w(w^\sigma - z)}\right] \\ &= \sigma\left[\sum_{\nu=0}^{p-1}\frac{z^\nu}{1-\sigma\nu} - z^p - z^{p+1}\int_N\frac{dw}{w(w^\sigma - z)}\right]. \end{aligned}$$

Upon expanding $[w(w^\sigma - z)]^{-1}$ in ascending powers of $1/z$, supposing for the moment that $|z| < 1$, we obtain

$$\begin{aligned} -z^{p+1}\int_N\frac{dw}{w(w^\sigma - z)} &= -pz^p\sum_{\nu=0}^{\infty}\frac{z^{\nu+1}}{\nu+1} = pz^p\log(1-z) \\ &= pi\pi z^p + pz^p\log(z-1). \end{aligned}$$

Whence, under the present hypotheses relation (55) becomes replaced by

$$S(z) = \sum_{\nu=0}^{p-1}\frac{z^\nu}{p-\nu} - \sigma z^p + \pi i z^p + z^p\log(z-1)$$

or, since

$$\begin{aligned} z^p\log(z-1) &= z^p\log z + z^p\log\left(1-\frac{1}{z}\right) = z^p\log z - \sum_{\nu=0}^{\infty}\frac{1}{(\nu+1)z^{\nu-p+1}} \\ &= z^p\log z - \sum_{\nu=0}^{p-1}\frac{z^\nu}{p-\nu} - \sum_{\nu=1}^{\infty}\frac{1}{(p+\nu)z^\nu}; \qquad |z| > 1, \end{aligned}$$

we may write when $|z| > 1$

$$(66)\qquad S(z) = -\frac{z^p}{p} + \pi i z^p + z^p\log z + r(z),$$

where $r(z)$ is an expression developable asymptotically in the form (56).

The form (66) for $S(z)$ is then that which corresponds in the present case to (55)—*i. e.*, it holds for values of z confined to any region T_1 sufficiently remote from the origin throughout which *real part* $z^p > 1$, *imag part* $z^p < 0$. The corresponding form for regions T_2 in which *real part* $z^p > 1$, *imag part* $z^p > 0$ is obtained (cf. (44), (45)) by adding $2\pi i z^p$ to the right member of (66). Thus, instead of (57) we reach in the present case

$$(67)\quad F(z) \sim \frac{2\sin\pi z^p}{\sqrt[2p]{2\pi z^p}}\exp\left[\theta\pi i z^p + \sum_{\nu=1}^{p-1}\zeta\left(\frac{\nu}{p}\right)\frac{z^\nu}{\nu} + \frac{z^p}{p}(c-1) + z^p\log z\right],$$

where $\theta = 0$ or $\theta = 1$ according as z is confined to T_1 or T_2.

Upon observing that when *imag part* $z^p > 0$ the function $\exp \pi i z^p$ is developable asymptotically in the form (56) with $a_0 = a_1 = a_2 = \cdots = 0$, it appears

that (58) becomes replaced in the present case by

$$F(z) \sim \frac{2 \sin \pi z^p}{\sqrt[2p]{2\pi z^p}} \exp \left[\sum_{\nu=1}^{p-1} \zeta\left(\frac{\nu}{p}\right) \frac{z^\nu}{\nu} + \frac{z^p}{p}(c-1) + z^p \log z \right].$$

This relation holds true, then, whenever *real part* $z^p > 1$. In case *real part* $z^p < 1$ the equations corresponding to (59) and (60) become respectively

$$H(z) = -\frac{1}{2p} \log 2\pi - \frac{1}{2p} \log (-z)^p - \tfrac{1}{2} \log\left(1 - \frac{1}{z}\right) - \Omega_z(1) + \pi i z^p$$

$$+ \sum_{\nu=1}^{p-1} \zeta\left(\frac{\nu}{p}\right) \frac{z^\nu}{\nu^p} + \frac{z^p}{p}(c-1) + z^p \log z + r(z),$$

$$F(z) \sim \frac{1}{\sqrt[2p]{2\pi(-z)^p}} \exp \left[\sum_{\nu=1}^{p-1} \zeta\left(\frac{\nu}{p}\right) \frac{z^\nu}{p} + \frac{z^p}{p}(c-1) + z^p \log(-z) \right].$$

Finally, in case z^p is real and positive, *i. e.*, in case φ has one of the arguments

$$0, \; -\frac{2\pi}{p}, \; -\frac{4\pi}{p}, \; -\frac{6\pi}{p}, \; \cdots,$$

we find by reasoning analogous to that at the close of § 24 that $S(z)$ will be given by (66) and hence we shall have (67) in which $\theta = 0$.

In summary we arrive then at the following

THEOREM II.[22] *Given the typical integral function $F(z)$ defined in Theorem I together with the assumption, that $\rho =$ the integer p.*

For values of z of large modulus lying within sectors of the type

$$-\frac{4k+3}{2p}\pi < \varphi < -\frac{4k+1}{2p}\pi \qquad \begin{cases} k = 0, 1, 2, 3, \cdots \\ -2\pi < \varphi < 0; \end{cases} \qquad \varphi = \arg z$$

we shall then have

$$F(z) \sim \frac{1}{\sqrt[2p]{2\pi(-z)^p}} \exp \left[\sum_{\nu=1}^{p-1} \zeta\left(\frac{\nu}{p}\right) \frac{z^\nu}{\nu} + \frac{z^p}{p}(c-1) + z^p \log(-z) \right]$$

ζ being the symbol for the Riemann ζ function, and c representing Euler's constant, while for values of z of large modulus lying within sectors of the type

$$-\frac{4k+1}{2p}\pi < \varphi < -\frac{4k-1}{2p}\pi \qquad \begin{cases} k = 0, 1, 2, 3, \cdots \\ -2\pi < \varphi \leqq 0; \end{cases} \qquad \varphi = \arg z$$

we shall have

$$F(z) \sim \frac{2 \sin \pi z^p}{\sqrt[2p]{2\pi z^p}} \exp \left[\sum_{\nu=1}^{p-1} \zeta\left(\frac{\nu}{p}\right) \frac{z^\nu}{\nu} + \frac{z^p}{p}(c-1) + z^p \log z \right].$$

[22] Cf. MATTSON (*l. c.*), pp. 15–17.

27. It will be observed that the integral function $F(z)$ considered in §§ 24–26 is of *order* > 0. BARNES[23] has also considered the corresponding problem for certain type functions whose order is *equal* to zero, but we shall confine ourselves to the case treated above.

ASYMPTOTIC DEVELOPMENTS OF FUNCTIONS DEFINED BY POWER SERIES

28. The results thus far indicated in the present chapter are but indirectly applicable to the determination of asymptotic developments for functions defined by power series. This subject, however, is one of evident importance. We shall now point out a general theorem in this field, resulting from the lemma of § 24.

THEOREM III. *If the coefficient $g(n)$ of the power series*

$$\sum_{n=0}^{\infty} g(n)z^n; \qquad r = \textit{radius of convergence} > 0, \tag{68}$$

may be considered as a function $g(w)$ of the complex variable $w = x + iy$ and as such satisfies the following conditions: (a) is single valued and analytic throughout the finite w plane except for a finite number of singularities situated at the points $w = w_1, w_2, \cdots, w_p$, none of which coincide with one of the points $w = 0, 1, 2, 3, \cdots$, and (b) is such that to an arbitrarily small positive constant ϵ there corresponds a positive constant K (independent of x and y) such that

$$\left|\frac{g(x \pm iy)}{g(x)}\right| < K \exp \epsilon y$$

for all values (real) of x and for all positive values of y sufficiently large, then the function $f(z)$ defined by (68) ($|z| < r$) will be such that for all values of z lying in any sector (center at $z = 0$) that does not include the positive real axis we may write

$$f(z) \sim -\sum_{m=1}^{p} r_m - \frac{g(-1)}{z} - \frac{g(-2)}{z^2} - \frac{g(-3)}{z^3} - \cdots, \tag{69}$$

where r_m represents the residue of the function

$$\frac{\pi g(w)(-z)^w}{\sin \pi w} \tag{70}$$

at the point $w = w_m$.

In order to prove this theorem we observe that for all values of z except those real and positive we may at once apply the lemma of § 24 with a taken as an arbitrarily large negative integer: $a = -l$, and write

$$\sum_{n=-l}^{\infty} g(n)z^n = \sum_{n=-l}^{-1} g(n)z^n + f(z) = -\sum_{m=1}^{p} r_m + \epsilon_l(z), \tag{71}$$

[23] See *Philosophical Trans.*, Vol. 199A (1902), pp. 466–468.

where $\epsilon_l(z)$ vanishes to as high an order as the $(l + \frac{1}{2})$th when $|z| = \infty$. Whence follows the indicated result.

For example, let us consider the function

$$(72) \qquad f(z) = \sum_{n=0}^{\infty} \frac{z^n}{\rho - n}; \qquad \rho \neq \textit{integer} < 0.$$

Here we may take

$$g(w) = \frac{1}{\rho - w}$$

and the residue of (70) at the pole $w = \rho$ is readily found to be

$$r_1 = \frac{\pi(-z)^\rho}{\sin \pi\rho}.$$

Whence, throughout any sector such as indicated in the above theorem we shall have

$$(73) \qquad f(z) \sim -\frac{\pi(-z)^\rho}{\sin \pi\rho} - \frac{1}{(\rho+1)z} - \frac{1}{(\rho+2)z^2} - \cdots.^{24}$$

This result ceases to hold when $\rho =$ a negative integer since the expression then has a pole of the second order at $w = \rho$. Such cases may, however, be treated by the same theorem. Thus, in particular, when $\rho = -1$ we obtain directly

$$r_1 = \frac{\log(-z)}{-z}$$

and hence, instead of (73)

$$(74) \qquad f(z) \sim \frac{\log(-z)}{z} - \frac{1}{z^2} - \frac{1}{2z^3} - \frac{1}{3z^4} - \cdots.$$

This result may be verified by noting that when $\rho = -1$ the equation (71) gives

$$f(z) = \frac{\log(1 - z)}{z}$$

while the power series appearing in (74) converges when $|z| > 1$ to the value $\frac{1}{z}\log(1 - 1/z)$ so that (74) gives the same form for $f(z)$.

29. *Generalizations of Theorem III.*—If for a given series (68) the function $g(w)$ is not single valued throughout the w plane, but contains q branch points $w = \bar{w}_1, \bar{w}_2, \cdots, \bar{w}_q$, conditions ($a$) and ($b$) remaining otherwise the same, the theorem continues true provided that, after rendering $g(w)$ single valued by means of q cuts extending vertically downwards[25] to infinity from the points

[24] Since the series here appearing is convergent for $|z| > 1$ the symbol $\sim$ may be changed to $=$.

[25] More generally, in any direction tending to infinity in the right half of the plane or vertically upwards or downwards.

$w = \bar{w}_m$; $(m = 1, 2, \cdots, q)$ respectively, we subtract from the second member of (69) the expression

$$\sum_{m=1}^{q} a_m$$

where a_m represents the loop integral (assumed to exist) of

$$\frac{1}{2i} \frac{g(w)(-z)^w}{\sin \pi w}$$

taken in the positive sense from the point $w = \bar{w}_m - i\infty$ to the same point after surrounding the (one) branch point $w = \bar{w}_m$. This result, in fact, appears directly upon reference to the demonstration of the theorem.

We note that in case the point $w = \bar{w}_m$ coincides with a point of the type $w = w_m$ mentioned in the theorem, the corresponding value of r_m is to be neglected, the term a_m then being evidently the only one of the two to be retained.

A particular type of function $f(z)$ to which Theorem III and these supplementary remarks apply is the following, discussed by BARNES:[26]

$$f_\beta(z; \theta) = \sum_{n=0}^{\infty} \frac{z^n \chi(n+\theta)}{(n+\theta)^\beta}; \qquad \begin{array}{l} \theta = \mathit{constant} \neq 0 \text{ or neg. integer}, \\ \beta = \mathit{constant}, \end{array}$$

where $\chi(1/z)$ is regular at the origin. Besides this, BARNES considers the corresponding problem for certain special types of functions for which condition (b) of Theorem III is *not* fulfilled. Of these latter may be especially mentioned the function

$$F_\beta(z; \theta) = \sum_{n=0}^{\infty} \frac{z^n}{(n+\theta)^\beta \Gamma(n+1)}; \qquad \begin{array}{l} \theta = \mathit{constant} \neq 0 \text{ or neg. integer}, \\ \beta = \mathit{constant}, \end{array}$$

for which it is stated[27] that for all values of z of large modulus we may write

$$F_\beta(z; \theta) \sim \varphi_\beta(z, \theta) + e^z z^{-\beta} \left[a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \right],$$

where $\varphi_\beta(z; \theta)$ represents the loop integral of the function

$$-\frac{1}{2\pi i} \frac{z^s \Gamma(-s)}{(s+\theta)^\beta}$$

taken over the path in the s plane extending from the point $s = -\infty + \mathit{imag\ part}\ (-\theta)$ to the point $s = -\theta$ and return.[28] The values of $a_0, a_1, a_2, \cdots$, are also given.

This function $F_\beta(z; \theta)$ typifies an important class of functions, *viz.*, those which for an appropriate value of *arg* z become infinite like $e^z z^k$ $(k = \mathit{const.})$

[26] *Philosophical Transactions*, Vol. 206A (1906), pp. 257, 272, 282.

[27] *L. c.*, p. 265.

[28] BARNES examines in further detail the properties of this loop integral, expressing it in the form of a series in his final result (p. 265). Cf. also *Quarterly Journ. of Math.*, Vol. 37 (1906), p. 89 *et seq*; also *ibid.*, Vol. 38, p. 116 *et seq.*

when $|z|$ is large. In this connection the following more general statement seems probable,[29] though a rigorous proof of it cannot be supplied by the author at present.

"If the function $g(n)$ appearing in the coefficients of the power series

$$f(z) = \sum_{n=0}^{\infty} \frac{g(n)}{\Gamma(n+1)} z^n$$

may be considered as a function $g(w)$ of the complex variable $w = x + iy$ and as such satisfies the following two conditions (a) is single valued and analytic throughout the finite w plane except for a finite number of singularities situated at the points $w = w_1, w_2, \cdots, w_p$ (none of which, however, coincide with one of the points $w = 0, 1, 2, 3, \cdots$) and (b) is such that there exists a constant (real or complex) β for which the function

$$G(w) = \frac{-\Gamma(-w)}{\Gamma(1-w-\beta)} g(w)$$

is developable in the form

$$\begin{aligned} G(w) = \frac{b_0}{w+\beta} + \frac{b_1}{(w+\beta)(w+\beta+1)} + \frac{b_2}{(w+\beta)(w+\beta+1)(w+\beta+2)} \\ + \cdots + \frac{b_n + \epsilon_n(w)}{(w+\beta)(w+\beta+1) + \cdots (w+\beta+n)}, \end{aligned} \tag{75}$$

where

$$\lim_{|w|=\infty} \epsilon_n(w) = 0; \qquad -\frac{\pi}{2} \leqq arg\ w \leqq \frac{\pi}{2},$$

then, for all values of z of large modulus we may write

$$f(z) \sim -\sum_{m=1}^{p} r_m + (-z)^{\beta} e^z \left[b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots \right],$$

in which r_m represents the residue of the function

$$\frac{\pi g(w)(-z)^w}{\Gamma(w+1) \sin \pi w} = -\Gamma(-w) g(w) (-z)^w$$

at the point $w = w_m$ and in which the coefficients $b_0, b_1, b_2, \cdots$, are determined from (75)."[30]

[29] From considerations based upon the relation

$$\frac{1}{2i} \int \frac{(-z)^w dw}{\Gamma(w+k) \sin \pi w} = \frac{e^z}{z^{k-1}} + p\left(\frac{1}{z}\right); \qquad k = constant \geqq 1,$$
$$p\left(\frac{1}{z}\right) = polynomial\ in\ \frac{1}{z},$$

where the indicated integration takes place in any closed (infinite) contour embracing the points $w = 0, 1, 2, 3, \cdots$.

[30] No mention has been made in the present Chapter of a class of power series whose asymptotic forms have been studied by DIENES and VALIRON. For a concise statement of their results see Theorems I and II in VALIRON's paper, "Sur le calcul approché de certaines fonctions entières," *Bull. de la Soc. Math. de France*, Vol. 42 (1914), pp. 252–264.

CHAPTER III

THE ASYMPTOTIC SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

30. The oldest and most fully developed aspect of the theory of asymptotic series concerns the so-called "asymptotic solutions" of linear differential equations. In the present chapter we shall undertake to give a summary of the principal results (without proofs) that have been obtained in this field, with indications as to certain noteworthy questions still remaining unanswered. Corresponding results and questions for linear difference equations will also be briefly considered.

Real Variable

31. Confining the attention at first to the case in which the independent variable x is real and positive, the investigations referred to may be said to cluster about the homogeneous linear differential equation

$$(1) \qquad y^{(n)} + a_1(x)y^{(n-1)} + a_2(x)y^{(n-2)} + \cdots + a_n(x)y = 0,$$

wherein the coefficients $a_1, a_2, \cdots, a_n$ are assumed to be developable for large positive values of x either in convergent or asymptotic series of the form

$$a_r(x) \sim x^{rk}\left[a_{r,0} + \frac{a_{r,1}}{x} + \frac{a_{r,2}}{x^2} + \cdots\right]; \qquad r = 1, 2, \cdots, n.$$

k being zero or some positive integer.[1] In this equation the point $x = \infty$ is in general a so-called "irregular point"[2] so that the usual "normal solutions" about the point $x = \infty$, as provided by the well-known theories of FUCHS, come to involve power series in $1/x$ that are *divergent* for all values of x.[3] Nevertheless, the same solutions continue to satisfy the equation formally[4] and it can be shown that they represent asymptotically, in the precise sense of § 13, certain *actual* solutions. In fact, we may begin by citing the following noteworthy theorem first established rigorously by HORN:[5]

"If for the equation (1) the roots $m_1, m_2, \cdots, m_n$ of the characteristic equation—*i. e.*, the algebraic equation

[1] The integer $k + 1$ is termed the *rank* of (1) at $x = \infty$. See for example HORN, "Gewöhnliche Differentialgleichungen belieber Ordnung" (Leipzig, Göschen, 1905), p. 187.

[2] For an exposition of the definitions and basal theorems in the theory of linear differential equations, one may consult PICARD's "Traité d'Analyse" (1896), Vol. 3, Chap. 11.

[3] Cf. PICARD, *l. c.*, § 22.

[4] Cf. PICARD, *l. c.*, § 23.

(2) $$m^n + a_{1,0}m^{n-1} + \cdots + a_{n,0} = 0,$$

are distinct from one another, equation (1) possesses n linearly independent solutions $y_1, y_2, \cdots, y_n$ valid for large positive values of x which are developable asymptotically in the forms

(2′) $$y_r \sim e^{f_r(x)}x^{\rho_r}\sum_{j=0}^{\infty}\frac{A_{r,j}}{x^j}; \qquad r = 1, 2, \cdots, n,$$

where $f_r(x)$ is a polynomial of degree $k+1$ in x, the coefficient of whose highest power in x is $m_r/(k+1)$, while ρ_r and $A_{r,j}$ are constants[6] with $A_{r,0} = 1$."[7]

If in this theorem the restriction be removed that the roots of the characteristic equation be distinct—*i. e.*, if multiple roots be present—the theorem fails and we at once encounter a problem for which no general solution has as yet been obtained. However, LOVE[8] has recently made a noteworthy advance in this direction, his theorem (which manifestly contains the above as a special case) being as follows:

If, other conditions remaining as stated above, "the characteristic equation has l roots $m_1, m_2, \cdots, m_l$, occurring respectively $n_1, n_2, \cdots, n_l$ times $(n_1 + n_2 + \cdots + n_l = n)$ and such that no multiple root of the characteristic equation is also a root of the equation

(3) $$a_{1,1}m^{n-1} + a_{2,1}m^{n-2} + \cdots + a_{n,1} = 0,$$

then the equation (1) possesses a fundamental system of solutions $y_{r,q}$ $(r = 1, 2, \cdots, l;\ q = 1, 2, \cdots, n_r)$ developable asymptotically in the form

$$y_{r,q} \sim e^{f_{r,q}(x)}\sum_{i=0}^{n_r-1}x^{\rho_{r,q}-i/n_r}\sum_{j=0}^{\infty}\frac{A_{r,q,i,j}}{x^j},$$

where $f_{r,q}(x)$ is a certain polynomial of degree $n_r(k+1)$ in x^{1/n_r}, the quantities $\rho_{r,q}$ and $A_{r,q,i,j}$ are determinate constants, and $A_{r,q,0,0} = 1$."

LOVE has furthermore considered in detail[9] the equations (1) of the second and third orders, including the cases in which (3) is satisfied by a multiple root,

[6] The precise values of the coefficients of $f_r(x)$ and of the constants ρ_r, $A_{r,j}$ may be determined by the method of undetermined coefficients after substituting y_r in (1). A similar remark should be understood with reference to the $f_{r,q}(x)$, $\rho_{r,q}$, etc., that follow.

[7] Historically, the first form of equation (1) to be studied in this connection was that taken by POINCARÉ in which $a_1, a_2, \cdots, a_n$ are rational fractions, thus possessing no other singularities than poles at $x = \infty$. See *Acta Math.*, Vol. 8 (1886), pp. 295–344.

[8] Cf. *Annals of Math.*, Vol. 15 (1914), p. 155.

[9] LOVE does not use, at least directly, the method common to the greater part of HORN's work, viz., that of successive approximations, though the latter could doubtless be employed to the same ends. His method rests rather upon certain general studies of DINI to be found in Vol. 2 (1898) of the *Annali di Mat.*, pp. 297–324, wherein the equation (1) of the nth order is first converted into a VOLTERRA integral equation of the second kind containing n arbitrary functions, termed "auxiliary functions," and the latter (equation) solved by the usual process of iteration, thus yielding forms of solution for the original equation (1). Through the arbitrariness existing

and has arrived at complete results for these orders.[10] Thus, for $n = 2$ we have the following:[11]

"In the differential equation

$$y'' + b(x)y = 0$$

suppose that $b(x)$ is a real or complex function developable asymptotically for large real positive values of x in the form

$$b(x) \sim x^{2k}\left[b_0 + \frac{b_1}{x} + \cdots\right],$$

where k is 0 or a positive integer. Then, for the same values of x equation possesses two linearly independent solutions y_1, y_2 such that (*a*) if $b_0 \neq 0$, *i. e.*, if the roots m_1, m_2 of the characteristic equation $m^2 + b_0 = 0$ are distinct, we may write

$$y_r \sim e^{f_r(x)}x^{\rho_r}\left[1 + \frac{A_{r,1}}{x} + \cdots\right], \qquad r = 1, 2,$$

where

$$f_r(x) = \frac{m_r x^{k+1}}{k+1} + \frac{\alpha_{r,-k}x^k}{k} + \cdots + \alpha_{r,-1}x;$$

(*b*) if $b_0 = 0$, $b_1 \neq 0$ we may write

$$y_r \sim e^{f_r(x)}x^{\rho_r}\left[1 + \frac{A_{r,1}}{x} + \cdots + \frac{1}{\sqrt{x}}\left(B_{r,0} + \frac{B_{r,1}}{x} + \cdots\right)\right], \qquad r = 1, 2,$$

where

$$f_r(x) = \frac{\alpha_{r,-2k-1}x^{k+\frac{1}{2}}}{k+\frac{1}{2}} + \frac{\alpha_{r,-2k}x^k}{k} + \cdots + \frac{\alpha_{r,-1}x^{\frac{1}{2}}}{\frac{1}{2}};$$

(*c*) if $k = b_0 = b_1 = 0$ we may write in general

$$y_r \sim x^{\rho_r}\left[1 + \frac{A_{r,1}}{x} + \cdots\right], \qquad r = 1, 2;$$

(*d*) but if $\rho_2 = \rho_1$ or, in general, if $\rho_2 - \rho_1$ is a positive integer we have[12]

in the choice of these auxiliary functions, the resulting solutions, though frequently complicated, are of great flexibility and it thus becomes possible to adapt them to a wide variety of investigations, as DINI himself has abundantly shown in a series of papers in the *Annali di Mat.* extending over the years 1898–1910. In the case of studies such as are being considered in the present chapter, the method readily provides *actual* solutions that are valid for large (positive) values of x and thus the problem becomes merely that of showing that the auxiliary functions may be chosen in particular in such a way that these solutions are developable asymptotically in the sense of § 13.

[10] Cf. *Am. Journ. of Math.*, Vol. 34 (1914), pp. 165–166.

[11] For the sake of completeness the case of unequal roots, though covered by the above mentioned theorems, is included in the statement.

[12] It will be observed that (*b*), (*c*) and (*d*) relate to the cases in which $m_1 = m_2$. If in (*c*) or (*d*) the series for $b(x)$ converges for all $|x| > R$ then $x = \infty$ is a "regular point" of the differential equation and hence in the results for y_1 and y_2 the sign $\sim$ may be changed to $=$; $|x| > R$.

$$y_1 \sim x^{\rho_1}\left[1 + \frac{A_{1,1}}{x} + \cdots\right],$$

$$y_2 \sim y_1 \log x + x^{\rho_2}\left[A_{2,0} + \frac{A_{2,1}}{x} + \cdots\right]."$$

The complete result for the equation of the third order is as follow

"In the differential equation

$$y''' + b(x)y' + c(x)y = 0$$

suppose that $b(x)$ and $c(x)$ are real or complex functions developab totically when x is large and positive in the forms

$$b(x) \sim x^{2k}\left[b_0 + \frac{b_1}{x} + \cdots\right],$$

$$c(x) \sim x^{3k}\left[c_0 + \frac{c_1}{x} + \cdots\right],$$

where k is 0 or a positive integer, and suppose that $b'(x)$ also has an a development. Then for the same values of x the given equation has thr independent solutions y_1, y_2, y_3 possessing asymptotic developments

(a) If the roots m_1, m_2, m_3 of the characteristic equation

$$m^3 + b_0 m + c_0 = 0$$

are distinct, we may write

$$y_r \sim e^{f_r(x)} x^{\rho_r}\left[1 + \frac{A_{r,1}}{x} + \cdots\right]; \qquad r = 1, 2, 3,$$

where

$$f_r(x) = \frac{m_r x^{k+1}}{k+1} + \frac{\alpha_{r,-k} x^k}{k} + \cdots + \alpha_{r,-1} x.$$

(b) If $m_1 \neq m_2 = m_3$ we may write in general

$$y_1 \sim e^{f_1(x)} x^{\rho_1}\left[1 + \frac{A_{1,1}}{x} + \cdots\right],$$

$$y_r \sim e^{f_r(x)} x^{\rho_r}\left[1 + \frac{A_{r,1}}{x} + \cdots + \frac{1}{\sqrt{x}}\left(B_{r,0} + \frac{B_{r,1}}{x} + \cdots\right)\right]; \qquad r =$$

where $f_1(x)$ has the same form as in (a) and

$$f_r(x) = \frac{m_r x^{k+1}}{k+1} + \frac{\alpha_{r,-2k-1} x^{k+\frac{1}{2}}}{k+\frac{1}{2}} + \frac{\alpha_{r,-2k} x^k}{k} + \cdots + \frac{\alpha_{r,-1} x^{\frac{1}{2}}}{\frac{1}{2}}; \qquad r =$$

(c) But if in (b) $\rho_3 = \rho_2$, or in general if $\rho_3 - \rho_2$ is a positive intege

$$y_r \sim e^{f_r(x)} x^{\rho_r}\left[1+\frac{A_{r,1}}{x}+\cdots\right]; \qquad r=1,2,$$

$$y_3 \sim y_2 \log x + e^{f_2(x)} x^{\rho_3}\left[A_{3,0}+\frac{A_{3,1}}{x}+\cdots\right],$$

where $f_1(x)$ and $f_2(x)$ have the same form as in (*a*).

(*d*) If $m_1 = m_2 = m_3$ and either $c_1 \neq 0$ or $b_1 = c_1 = 0$, $c_2 \neq 0$ we may write

$$y_r \sim e^{f_r(x)} x^{\rho_r}\left[1+\frac{A_{r,1}}{x}+\cdots+\frac{1}{x^{\frac{1}{3}}}\left(B_{r,0}+\frac{B_{r,1}}{x}+\cdots\right)\right.$$
$$\left.+\frac{1}{x^{\frac{2}{3}}}\left(C_{r,0}+\frac{C_{r,1}}{x}+\cdots\right)\right]; \qquad r=1,2,3,$$

where

$$f_r(x) = \frac{m_r x^{k+1}}{k+1}+\frac{\alpha_{r,-3k-2}x^{k+\frac{2}{3}}}{k+\frac{2}{3}}+\frac{\alpha_{r,-3k-1}x^{k+\frac{1}{3}}}{k+\frac{1}{3}}+\cdots+\frac{\alpha_{r,-1}x^{\frac{1}{3}}}{\frac{1}{3}}.$$

(*e*) If $c_1 = 0$, $b_1 \neq 0$, y_1, y_2, y_3 have expansions of the same form as in (*b*).

(*f*) If $k = b_1 = c_1 = c_2 = 0$ we may write

$$y_1 \sim x^{\rho_1}\left[1+\frac{A_{1,1}}{x}+\cdots\right],$$

$$y_2 \sim Ay_1 \log x + x^{\rho_2}\left[A_{2,0}+\frac{A_{2,1}}{x}+\cdots\right],$$

$$y_3 \sim By_1 \log^2 x + x^{\rho_2}\log x\left[B_{3,0}+\frac{B_{3,1}}{x}+\cdots\right]+x^{\rho_3}\left[A_{3,0}+\frac{A_{3,1}}{x}+\cdots\right].\text{''}[13]$$

While the complete results for the equation (1) of order $n \geqq 4$ have not as yet been obtained, a careful examination of those just given for $n = 2, 3$ throws light upon what the corresponding forms may be expected to be. Moreover, in connection with this question the following result should be noted:[14]

"Let $\tau(x)$ be one of the system of functions

$$(3)' \qquad \frac{1}{x^{1+\nu}},\quad \frac{1}{x(\log x)^{1+\nu}},\quad \frac{1}{x\log x(\log\log x)^{1+\nu}},\quad \cdots,\quad (\nu > 0)$$

and put

$$\tau_1(x) = \int_x^\infty \tau(x)dx.$$

[13] It will be observed that (*b*) and (*c*) refer to the case $m_1 \neq m_2 = m_3$ while (*d*), (*e*) and (*f*) refer to the case $m_1 = m_2 = m_3$. If in (*f*) the series for $b(x)$ and $c(x)$ converge for all $|x| > R$ then $x = \infty$ is a regular point of the differential equation and hence in the results for y_1, y_2 and y_3 the ∞ may be changed to $=$; $|x| > R$.

[14] Obtained by Dini for the case in which the roots of the characteristic equation all have the same *real part*, and partially obtained by him when this restriction is removed (*Annali di Mat.*, Vol. 3 (1899), p. 136. The result has recently been established in its entirety by Love in the *American Journal of Mathematics*.

Suppose now that in the differential equation

(4) $y^{(n)} + [a_1 + \alpha_1(x)]y^{(n-1)} + [a_2 + \alpha_2(x)]y^{(n-2)} + \cdots + [a_n + \alpha_n(y$

the functions $\alpha_1(x)$, $\alpha_2(x)$, $\cdots$, $\alpha_n(x)$ together with their $2n - 1$ deriv continuous when x is sufficiently large, and suppose that the cha equation

$$(5) \qquad \mu^n + a_1\mu^{n-1} + \cdots + a_n = 0$$

has l different roots μ_1, μ_2, $\cdots$, μ_l occurring n_1, n_2, $\cdots$, n_l times re $(n_1 + n_2 + \cdots + n_l = n)$ and let n' be the largest of the numbers n_1,

If one of the functions $\tau(x)$ exists such that for sufficiently large v

$$(6) \qquad |\alpha_r^{(s)}(x)| \leqq \frac{\tau(x)}{x^{2n'-2}}; \qquad r = 1, 2, \cdots, n; \qquad s = 0, 1, \cdots, 2n -$$

then for the same values of x the equation (4) has n linearly indepen tions $y_{i,k}(x)$ expressible in the form

$$y_{i,k}(x) = x^{k-1}e^{\mu_i x}[1 + \epsilon_{i,k}(x)]; \qquad i = 1, 2, \cdots, l; \qquad k = 1, 2, \cdot$$

where $\epsilon_{i,k}(x)$ vanishes at infinity to at least as high an order as tha while further

$$y_{i,k}^{(s)}(x) = x^{k-1}e^{\mu_i x}[\mu_i^s + \zeta_{i,k,s}(x)]; \qquad s = 1, 2, \cdots, n,$$

where $\lim\limits_{x=\infty} \zeta = 0$."

It is to be observed that for the special case $\tau(x) = 1/x^2$ this res to an equation of the form (1) (wherein $k = 0$), and furnishes the " terms" of developments for the corresponding solutions $y_{i,k}(x)$. by a sufficiently critical examination of the form of $\epsilon_{i,k}(x)$, these dev could be identified with asymptotic developments in the precise sens For the type of equation considered, the result is seen to be in every ser so far as the possibility of multiple roots in (5) is concerned, excep restrictions (6). These latter when interpreted with reference to (1)

$$(7) \qquad a_{r,s} = 0; \qquad r = 1, 2, 3, \cdots, n; \qquad s = 1, 2, 3, \cdots, 2n' -$$

and hence come to impose unfortunate restrictions. However, the r decided value in showing that all further studies upon the problen may be limited to those cases (assuming multiple roots present in (2 (7) are *not* satisfied.

Complex Variable

32. Passing to the corresponding studies upon (1) when the in variable x is allowed to take on complex values, the existence, form of the asymptotic solutions have been completely discussed by Br case the coefficients $a_r(x)$ $(r = 1, 2, \cdots, n)$ are developable in conver

($|x| > R =$ constant sufficiently large) and under the assumption that the roots of the characteristic equation (2) are distinct.[15] Corresponding results when multiple roots are present in (2) do not appear to have been thus far obtained.

BIRKHOFF's essential result may be summarized as follows:

"Representing by $m_1, m_2, \cdots, m_n$ the n (distinct) roots of (2), let there be drawn from the origin ($z = 0$) the $N = n(n-1)(k+1)$ rays ("critical" rays) determined by the equation

$$\textit{real part of } [(m_s - m_t)x^{k+1}] = 0; \qquad s \neq t.$$

Let the angles which these rays make with the positive real axis in the order of their increasing magnitude be denoted by $\tau_1, \tau_2, \cdots, \tau_N$ and place $\tau_{N+1} = \tau_1 + 2\pi$.

Then, corresponding to the sector $\tau_m \leqq \arg x < \tau_{m+1}$ there exists a set of fundamental solutions y_r ($r = 1, 2, \cdots, n$) of (1) developable asymptotically in the forms (2)′ where $f_r(x)$, ρ_r and $A_{r,j}$ continue to have the meanings there indicated.

The set of solutions satisfying (2)′ in the sector (τ_m, τ_{m+1}) differs at most by one solution from the set satisfying (2)′ in the adjacent sector (τ_{m+1}, τ_{m+2})."[16]

Linear Difference Equations

33. If instead of (1) we take for consideration the linear difference equation

$$(8) \quad y(x+h) + a_1(x)y(x+h-1) + a_2(x)y(x+h-2) + \cdots + a_n(x)y(x) = 0$$

wherein the coefficients $a_1, a_2, \cdots, a_n$ are assumed to be developable for large positive values of x either in convergent series or asymptotically in the forms

$$(8)' \qquad a_r(x) \sim x^{rk}\left[a_{r,0} + \frac{a_{r,1}}{x} + \frac{a_{r,2}}{x^2} + \cdots\right]; \qquad r = 1, 2, \cdots, n,$$

[15] *Trans. Am. Math. Soc.*, Vol. 10 (1909), pp. 463–468. BIRKHOFF considers, instead of (1), the system of n ordinary linear equations of the first order:

$$(A) \qquad \frac{dy_i}{dx} = \sum_{j=1}^{n} a_{ij}(x)y_j \qquad (i = 1, 2, \cdots, n),$$

in which for $|x| > R$ we have

$$a_{ij}(x) = a_{ij}x^q + a_{i,j}^{(1)}x^{q-1} + \cdots + a_{ij}^{(q)} + a_{ij}^{(q+1)}\frac{1}{x} + \cdots \quad (i, j = 1, 2, \cdots, n),$$

the characteristic equation then becoming

$$|a_{ij} - \delta_{ij}\alpha| = 0; \quad \delta_{ij} = 0 \ \text{ if } \ i \neq j; \quad \delta_{ij} = 1 \ \text{ if } \ i = j.$$

The equation (1) may be transformed into a system of the form (A) by placing $y_1 = x^{nk}y$, $y_2 = x^{(n-1)k}y'$, $\cdots$, $y_n = x^k y^{(n-1)}$ in which case we find $q = k$. Thus, whatever applies to (A) applies to (1) as a special case with $q = k$.

The important case in which the coefficients $a_r(x)$ of (1) are rational polynomials was discussed in a series of earlier papers by HORN whose results are summarized by VAN VLECK in the Boston Colloquium Lectures (1905), pp. 85–92.

[16] For the precise nature of this dependence, see BIRKHOFF, *l. c.*, p. 468.

k being zero or a positive integer, we have, corresponding to the first result cited in § 31, the following:

"If the roots $m_1, m_2, \cdots, m_n$ of the characteristic equation

$$m^n + a_{1,0}m^{n-1} + \cdots + a_{n,0} = 0 \tag{9}$$

are distinct and no one of them equal to zero, equation (8) possesses n linearly independent solutions $y_1, y_2, \cdots, y_n$ valid for large positive values of x which are developable asymptotically in the forms

$$y_r \sim [\Gamma(x+1)]^k m_r^x x^{\rho_r} \sum_{j=0}^{\infty} \frac{A_{r,j}}{x^j}; \qquad r = 1, 2, \cdots, n, \tag{9$'$}$$

where $A_{r,0} = 1$."[17]

In case (9) has multiple roots, or a zero root ($a_{n,0} = 0$) the principal results thus far obtained appear to be those of Nörlund who employs asymptotic "faculty series" and allows the independent variable x to range over complex as well as real values. Using his notation and including for the sake of completeness the case of distinct roots, his results are as follows:[18]

"Given the linear difference equation

$$\sum_{i=0}^{k} P_i(x)u(x-i) = 0, \tag{10}$$

where the coefficients are faculty series of the form

$$\begin{aligned} P_i(x) = c_0^{(i)} + \frac{c_1^{(i)}}{x+1} + \frac{c_2^{(i)}}{(x+1)(x+2)} \\ + \frac{c_3^{(i)}}{(x+1)(x+2)(x+3)} + \cdots; \qquad i = 0, 1, 2, \cdots, k \end{aligned} \tag{11}$$

all of which converge throughout the right half of the x plane.[19] Suppose first that the roots $a_1, a_2, a_3, \cdots, a_k$ of the characteristic equation

$$c_0^{(0)}z^k + c_0^{(1)}z^{k-1} + \cdots + c_0^{(k)} = 0; \qquad c_0^{(0)} \neq 0, \qquad c_0^{(k)} \neq 0 \tag{12}$$

are distinct. Then there exist k solutions $u_1, u_2, \cdots, u_k$ of (10) such that throughout the sector $-(\pi/2) + \epsilon < arg\, x < (\pi/2) - \epsilon$ (ϵ arbitrarily small and > 0) we have

$$u_j \sim a_j^x \frac{\Gamma(x+1)}{\Gamma(x - \rho_j + 1)} \varphi_j(x), \tag{13}$$

where ρ_j is a constant and $\varphi_j(x)$ a faculty series of the form indicated in (11).

[17] Cf. Horn, *Journ. für Math.*, Vol. 138 (1910), p. 159.

[18] "Kongelige Danske Videnskabernes Selskabs Skrifter" (Mém. de l'Acad. Roy. des Sciences et des Lettres de Denmark), Vol. 6 (1911), pp. 317–318. It would appear that the proofs of the results here stated have not as yet been published except in part.

[19] A very broad class of series of the form (11) have this property. See for example Nielson, "Handbuch der Theorie der Gammafunction," Leipzig (Teubner), 1906, § 96.

In case (12) has multiple roots and $a_j{}^x$ is an n-fold root, NÖRLUND distinguishes two cases:

(1) a_j is at the same time an $(n-p)$-fold root of the equations

$$\sum_{s=0}^{k} c_p{}^{(s)} z^{k-s} = 0; \qquad p = 1, 2, \cdots, n-1.$$

(2) These conditions are not fulfilled.

In (2) no asymptotic development exists of the form (13).

In (1) there exist n linearly independent solutions $u_s(x)$; $s = 1, 2, \cdots, n$ such that when $-(\pi/2) + \epsilon < arg\ x < (\pi/2) - \epsilon$ we have

$$u_s \sim a_j{}^x \Phi_s(x); \qquad s = 1, 2, 3, \cdots, n,$$

where

$$\Phi_s(x) = \varphi_0(x) \frac{\Gamma(x+1)}{\Gamma(x-\rho_s+1)} + \varphi_1(x) \frac{\partial}{\partial \rho_s} \frac{\Gamma(x+1)}{\Gamma(x-\rho_s+1)} + \cdots + \varphi_n(x) \frac{\partial^n}{\partial \rho_s{}^n} \frac{\Gamma(x+1)}{\Gamma(x-\rho_s+1)},$$

the expressions $\varphi_0, \varphi_1, \cdots, \varphi_n$ being developments of the form (11).

If some of the roots of (12) are zero or infinite, it is necessary in order to obtain a system of fundamental solutions to use a series of substitutions of the form

$$u(x) = [\Gamma(x)]^{\mu_r} u^{(\mu_r)}(x) = \Gamma^{\mu_r}(x) u^{(\mu_r)}(x)$$

and determine μ_r so that the difference equation in $u^{(\mu_r)}(x)$ shall have a characteristic equation containing at least one root which is finite and different from zero. It is always possible to determine in but one way a series of numbers $\mu_1, \mu_2, \cdots, \mu_m$ such that the total number of roots which are finite and different from zero in the corresponding characteristic equations thus obtained is exactly the order k of (10).[20] If, whenever a multiple root occurs in one of these characteristic equations, the corresponding conditions under (1) are satisfied, then there exists a system of fundamental solutions of (10) each of which is asymptotically represented within the sector $-(\pi/2) + \epsilon < arg\ x < (\pi/2) - \epsilon$ by a series of the form

$$\Gamma^{\mu_r}(x) a_j{}^x \Phi_s(x).$$

Exceptions occur, however, when some of the numbers μ_r are not integers, since the coefficients in the above-mentioned difference equations are then no longer developable in faculty series of the form (11). For example, suppose μ_r = a rational fraction p/q. We may then put $x = pz$, $u(x) = v(z)$ and derive from (10) a difference equation for $v(z)$, thus demonstrating the existence of solutions expressible asymptotically in the forms

$$\Gamma^{\mu_r}\left(\frac{x}{p}\right) a_j{}^{x/p} \Phi_s\left(\frac{x}{p}\right).$$

[20] NÖRLUND, *Acta Math.*, Vol. 34 (1911), p. 16.

Important studies of (8) when x is complex and under the assu
the roots of (9) are different from zero and distinct and that th
$a_r(x)$ are rational fractions developable in the forms (8)′ (wherein the
then converge for all $|x|$ sufficiently large), have been made also b
and by BIRKHOFF,[22] with the essential result that there exists a syst
mental solutions $G(x) \equiv y_1, y_2, y_3, \cdots, y_n$ developable asymptot
respective forms (9)′ throughout the *right* half of the x plane, and
time there exists a second system $H(x) \equiv y_1, y_2, \cdots, y_n$ of fundame
developable likewise in the forms (9)′ but throughout the *left* half
Moreover, the elements of the system $G(x)$ when considered in
plane possess asymptotic developments other than (9)′ whose for
arg x passes through any one of certain radial directions ("secon
rays") lying in the second and third quadrants,[23] while similarly
of $H(x)$ when considered in the right half plane are developable a
in forms differing from (9)′ and changing as *arg* x passes through
directions situated in the first and fourth quadrants.

Returning again to the case in which x is regarded as real and
assuming further that it is confined to integral values, we have, cor
the last result stated in § 31, the following:[24]

"Let $\tau(x)$ be one of the system of functions (3)′ and put

$$\tau_1(x) = \sum_{x_1=x+1}^{\infty} \tau(x_1).$$

Suppose now there is given a difference equation

$$(14)\qquad [a_0 + \alpha_0(x)]y(x+n) + [a_1 + \alpha_1(x)]y(x+n-1) + \cdots + [a_n + \alpha_n$$

whose characteristic equation

$$a_0\mu^n + a_1\mu^{n-1} + \cdots + a_n = 0$$

has l different roots $\mu_1, \mu_2, \cdots, \mu_n$ occurring $n_1, n_2, \cdots, n_l$ times
$(n_1 + n_2 + \cdots + n_l = n)$ and let n' be the largest of the numbers

[21] *Acta Math.*, Vol. 36 (1913), pp. 1–68; also *Compt. Rend.*, Vol. 148 (1909),

[22] *Trans. Am. Math. Soc.*, Vol. 12 (1911), pp. 243–284. As in his studies on li
equations (cf. footnote, p. —), BIRKHOFF considers a system of linear difference
first order. In order to identify the forms (9)′ with those occurring in his resu
observe that

$$\Gamma(x+1) \sim x^{x+1/2}e^{-x}\left(c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots\right).$$

See for example HORN, *Math. Annalen*, Vol. 53 (1900), p. 191.

[23] For the precise statement, see BIRKHOFF, *l. c.*, p. 277–278. See also p. 27

[24] Cf. LOVE in *Am. Journ. Math.* Obtained earlier by FORD in case all root
teristic equation have the same modulus (*Annali di Mat.*, Vol. 13 (1907), p. 328

If a function $\tau(x)$ exists such that for sufficiently large values of x

$$|\alpha_r(x)| \leqq \frac{\tau(x)}{x^{2n'-2}}; \qquad r = 0, 1, 2, \cdots, n,$$

then for the same values of x the equation (14) has n linearly independent solutions $y_{i,\,k}(x)$ expressible asymptotically in the forms

$$y_{i,\,k}(x) \sim x^{k-1}\mu_i{}^x[1 + \epsilon_{i,\,k}(x)]; \qquad i = 1, 2, \cdots, l; \qquad k = 1, 2, \cdots, n_i,$$

where $\epsilon_{i,\,k}$ vanishes at infinity to at least as high an order as that of $\tau_1(x)$."

Summary

34. A comparison of the results noted in §§ 30–33 would indicate that the study of the asymptotic solutions of either the differential equation (1) or the difference equation (8) is already in a fairly satisfactory state provided the assumption be made throughout that the roots of the characteristic equation are distinct, but much remains to be done in those cases where multiple roots are present. In fact, it is only for the equation (1) of the special orders $n = 2$ or $n = 3$ that we find what could be described as a complete discussion, and even this has thus far been carried out only for the *real* variable x.

CHAPTER IV

ELEMENTARY STUDIES ON THE SUMMABILITY OF SERIES

35. *Introduction.*—The divergent series

$$(1) \qquad 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

was regarded by EULER[1] as having the sum $\frac{1}{2}$ on the ground that the e
$1/(1 + x)$ gives rise by division to the series

$$(2) \qquad 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots,$$

so that in particular (placing $x = 1$) one must have

$$(3) \qquad \tfrac{1}{2} = 1 - 1 + 1 - 1 + 1 - 1 + \cdots.$$

In general, the "sum" of a series (convergent or divergent) was taken
number most naturally associated with it from the standpoint of mat
operations. This conception, however, naturally led to inconsistenc
by developing the expression $(1 - x^n)/(1 - x^m)$ into the form

$$(4) \qquad 1 - x^n + x^m - x^{n+m} + x^{2m} - \cdots,$$

and noting the result when $x = 1$ we obtain for the series (1) the sum $n/$
of $\frac{1}{2}$.

The notion of sum as thus loosely conceived was eventually replac
exact definition of ABEL and CAUCHY according to which the sum of a

$$(5) \qquad a_0 + a_1 + a_2 + a_3 + \cdots$$

is taken to mean the limit

$$(6) \qquad s = \lim_{n=\infty} (a_0 + a_1 + a_2 + \cdots + a_n).$$

Series for which this limit exists were termed *convergent,* all others *dive*

Of the two classes of series thus arising, the former occupied almost e
the attention of the immediate successors of ABEL and CAUCHY and t
extent that all divergent series came to be regarded as of questionable
indeed of doubtful significance. It is a noteworthy fact, however, t
and CAUCHY themselves never ceased to regard divergent series w
interest and with the belief that such series should by no means be bani
analysis for the mere reason that they fell outside the pale of the

[1] For a more extended historical account, see BOREL, "Leçons sur les Séries D

definition (6). Each felt on the other hand that the subject presented a rich field for further research.

Only since the time of WEIERSTRASS has the question thus arising—*viz.*, whether any numerical significance can properly be attached to a divergent series—been scientifically attacked and in large measure answered. The avenue of approach has been chiefly through the so-called boundary-value (Grenzwert) problem in the theory of analytic functions.[2] Thus, FROBENIUS[3] showed in the first place that if

$$\sum_{n=0}^{\infty} a_n x^n \tag{7}$$

be any power series having a radius of convergence equal to 1, then

$$\lim_{x=1-0} \sum_{n=0}^{\infty} a_n x^n = \lim_{n=\infty} \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1}, \tag{8}$$

where $s_n = a_0 + a_1 + a_2 + \cdots + a_n$. This was shown to be true, at least, whenever the limit indicated on the right exists. Now, the first member of (8) is naturally associated with the series (in general divergent)

$$\sum_{n=0}^{\infty} a_n, \tag{9}$$

so that it becomes natural to associate with the latter the sum

$$s = \lim_{n=\infty} \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1}, \tag{10}$$

whenever this limit exists. Formula (10), regarded as a general formula for defining the sum of any given divergent series (9), finds additional justification in the demonstrable fact that for any *convergent* series (9) the sum as defined by either (6) or (10) is the *same*—*i. e.*, formula (10) is *consistent*. Moreover, this selection for s is seen to bear an interesting relation to the early statement of EULER noted above respecting the particular series (1), since, when applied to (1), it gives at once $s = \frac{1}{2}$.

In the present chapter certain general studies are first undertaken (§§ 36–40) upon a few of the well-known, standard definitions for the "sum" of a divergent series. The definitions selected (which include (10) as a special case) are subjected in turn to a number of tests which it is believed any such definition may well be asked to satisfy, and the results attained are summarized in § 41.

[2] For a description of this problem see JAHRAUS, "Das Verhalten der Potenzreihen auf dem Konvergenzkreise historisch-kritisch dargestellt," Programm des Kgl. humanist. Gymnasiums Ludwigshafen a. Rhein (1901), pp. 1–56. See also KNOPP, "Grenzwerte von Reihen bei der Annäherung an die Konvergenzgrenze," Dissertation (Berlin, 1907).

[3] *Journ. für Math.*, Vol. 89 (1880), p. 262.

The underlying principles guiding the development of these §§ are
the Preface and hence need not be repeated here.

In the latter part of the chapter the essential properties of "a
summable" series are considered (§ 42) and this is followed by a fe
mentary theorems and remarks on the theory of summability in gener
being suppressed when reference can be readily made to them elsewhe

36. *Definitions of Sum.*—Let any given series (convergent or dive
represented by

$$(11) \qquad \sum_{n=0}^{\infty} u_n$$

and let us place

$$s_n = \sum_{n=0}^{n} u_n.$$

If (11) is convergent let its sum be indicated by S, if divergent let th
signed to it by whatever manner be indicated by s.

The definitions for s to which we shall confine our attention[4] are a

$$\text{(I)} \qquad s = \lim_{n=\infty} \frac{S_n^{(r)}}{D_n^{(r)}}; \qquad r = \textit{fixed integer} \geqq 0 \quad (\text{Cesàro}),^5$$

where

$$(12) \quad \begin{cases} S_n^{(r)} = s_n + rs_{n-1} + \dfrac{r(r+1)}{2!} s_{n-2} + \cdots + \dfrac{r(r+1) \cdots (r+}{n!} \\ D_n^{(r)} = \dfrac{(r+1)(r+2) \cdots (r+n)}{n!}. \end{cases}$$

Under (I) is thus included as a special case corresponding to $r = 1$ the
(10). The *least* value of r for which the second member of (I) exists is
degree of indeterminacy of the series (11).

[4] We have confined the attention to what may be called the older and best know
definition, (I) and (II) being connected with the early studies of Hölder and Cesàr
boundary value (Grenzwert) problem for functions defined by power series (see § 35
remainder, especially (III) and (IV), are connected with the independent and no
studies of Borel upon divergent series. A form of definition prominent in the
literature, especially in England, is that of Riesz (*Compt. Rend.*, July, 1909):

$$s = \lim_{n=\infty} \sum_{v=0}^{n-1} u_v \left(1 - \frac{v}{n}\right)^r; \qquad r = \textit{integer} \geqq 0.$$

There should be mentioned also the following definition of De La Vallée Poussin (
la classe des Sciences de l'Académie Royale de Belgique, 1908, pp. 193–254):

$$s = \lim_{n=\infty} \left(u_0 + \sum_{k=1}^{n} \frac{n(n-1) \cdots (n-k+1)}{(n+1)(n+2) \cdots (n+k)} u_k \right).$$

For a *general* study of possible forms of definition, see Silverman's Thesis "On the defi
sum of a divergent series" in the scientific publications of the University of Missou
1913, pp. 1–96.

[5] *Bulletin des Sciences Math.* (2), Vol. 14 (1890), p. 119. Chapman has extende
tion to include fractional values of r (*Proc. London Math. Soc.*, Vol. 9 (1911), pp. 36

(II) $$s = \lim_{n=\infty} s_n^{(r)}; \qquad r = \textit{fixed integer} \geqq 0 \quad (\text{Hölder}),$$[6]

where

$$\begin{cases} s_n^{(0)} = s_n, \\ s_n^{(1)} = \dfrac{1}{n+1}(s_0^{(0)} + s_1^{(0)} + \cdots + s_n^{(0)}), \\ s_n^{(2)} = \dfrac{1}{n+1}(s_0^{(1)} + s_1^{(1)} + \cdots + s_n^{(1)}), \\ \vdots \\ s_n^{(r)} = \dfrac{1}{n+1}(s_0^{(r-1)} + s_1^{(r-1)} + \cdots + s_n^{(r-1)}). \end{cases}$$

(III) $$s = \lim_{\alpha=+\infty} e^{-\alpha} s(\alpha) \quad (\text{Borel}),$$[7]

where $s(\alpha)$ is defined by the following series (assumed convergent for all values of α)

(13) $$s(\alpha) = \sum_{n=0}^{\infty} \frac{s_n}{n!} \alpha^n.$$

(IV) $$s = \int_0^{\infty} e^{-\alpha} u(\alpha) d\alpha \quad (\text{Borel}),$$[8]

where $u(\alpha)$ is defined by the following series (assumed convergent for all values of α)

(14) $$u(\alpha) = \sum_{n=0}^{\infty} \frac{u_n}{n!} \alpha^n.$$

(V) $$s = \int_0^{\infty} e^{-\alpha} u_p(\alpha) d\alpha; \qquad p = \textit{fixed integer} \geqq 1,$$

where

$$u_p(\alpha) = (u_0 + u_1 + \cdots + u_{p-1}) + (u_p + u_{p+1} + \cdots + u_{2p-1})\alpha \\ + (u_{2p} + \cdots + u_{3p-1})\alpha^2 + \cdots.$$

(VI) $$s = \int_0^{\infty} e^{-\alpha} U_p(\alpha) d\alpha,$$[9]

where

(15) $$U_p(\alpha) = \sum_{n=0}^{\infty} \frac{u_n \alpha^{np}}{(np)!}; \qquad p = \textit{fixed integer} \geqq 1.$$

37. *Consistency of the Above Definitions.*—It is at once to be assumed that any tenable definition of sum for divergent series must be such that in the case

[6] *Math. Annalen*, Vol. 20 (1882), pp. 535–549.
[7] Cf. "Leçons," p. 97.
[8] Cf. "Leçons," p. 98.
[9] Due to LeRoy. Cf. *Annales de la Faculté des Sciences de Toulouse* (2), Vol. 2 (1902), p. 217.

of a *convergent* series it gives $s = S$. This property of a definition is c
consistency.[10] We proceed to establish the consistency of all the above d
by a uniform method based upon the following general lemma in the t
limits.[11]

Lemma.—Let $s_0, s_1, s_2, \cdots, s_n, \cdots$ be a sequence of quantities (real
plex) such that $\lim_{n=\infty} s_n = l$ and let $a_0^{(p)}, a_1^{(p)}, a_2^{(p)}, \cdots, a_n^{(p)}, \cdots$ be a
of positive quantities (weights) dependent upon a parameter p (ind
of n). Also let it be supposed that the expression

$$S_p = \frac{\sum_{n=0}^{\infty} a_n{}^{(p)} s_n}{\sum_{n=0}^{\infty} a_n{}^{(p)}}$$

has a meaning for every value of p in a given sequence P of positive
which increase indefinitely to $+\infty$. If, then, p be allowed to inc
definitely ranging over the values in P we shall have $\lim_{p=+\infty} S_p = l$ provi

$$(A) \qquad \lim_{p=+\infty} \frac{\sum_{n=0}^{m} a_n{}^{(p)}}{\sum_{n=0}^{\infty} a_n{}^{(p)}} = 0,$$

where m is any fixed positive integer (independent of p and n).

Proof.—We have by hypothesis $s_n = l + \epsilon_n$; $\lim_{n=\infty} \epsilon_n = 0$ and it su
show that $\lim_{p=\infty} D_p = 0$, where

$$D_p = \frac{\sum_{n=0}^{\infty} a_n{}^{(p)} s_n}{\sum_{n=0}^{\infty} a_n} - l.$$

By writing

$$\sum_{n=0}^{\infty} a_n{}^{(p)} s_n = \sum_{n=0}^{m} a_n{}^{(p)} s_n + \sum_{n=m+1}^{\infty} a_n{}^{(p)} s_n$$

and then placing $s_n = l + \epsilon_n$ in the last term here appearing we obtain

$$D_p = \frac{\sum_{n=0}^{m} (s_n - l) a_n{}^{(p)} + \sum_{n=m+1}^{\infty} \epsilon_n a_n{}^{(p)}}{\sum_{n=0}^{\infty} a_n{}^{(p)}}.$$

[10] Cf. Bromwich, "Infinite Series" (London, 1908), § 100.

[11] Cf. Ford, *American Journ. of Math.*, Vol. 32 (1910), p. 320. As here gener
lemma was first obtained and applied to the discussions of the present chapter by Mr
in his thesis entitled "On the Characteristic Properties of Sum-Formulæ in the Theory of
Series," University of Michigan, 1911.

Whence, if we indicate by g_m a positive quantity such that $g_m \geqq |s_i|$; $i = 0, 1, 2, \cdots, m$, we may write

$$|D_p| \leqq (g_m + |l|) \frac{\sum_{n=0}^{m} a_n^{(p)}}{\sum_{n=0}^{\infty} a_n^{(p)}} + \frac{\sum_{n=m+1}^{\infty} |\epsilon_n| a_n^{(p)}}{\sum_{n=0}^{\infty} a_n^{(p)}}.$$

This relation holds good for any preassigned value of p belonging to P and for any preassigned arbitrarily large positive integral value of m. The same having been once established, let us now choose an arbitrarily small positive quantity ϵ and then take m so large that $|\epsilon_n| < \epsilon$; $n = m + 1, m + 2, \cdots$. We may then write

$$\sum_{n=m+1}^{\infty} |\epsilon_n| a_n^{(p)} < \epsilon \left[\sum_{n=0}^{\infty} a_n^{(p)} - \sum_{n=m}^{\infty} a_n^{(p)} \right].$$

Whence,

$$D_p < (g_m + |l|) \frac{\sum_{n=0}^{m} a_n^{(p)}}{\sum_{n=0}^{\infty} a_n^{(p)}} + \epsilon \left[1 - \frac{\sum_{n=0}^{m} a_n^{(p)}}{\sum_{n=0}^{\infty} a_n^{(p)}} \right],$$

from which the desired result follows as soon as we introduce the hypothesis (A).

38. We may now easily show the consistency of definition (I). For this purpose let us take P in the lemma of § 37 as the sequence of positive integers $0, 1, 2, 3, \cdots$, and let $a_n^{(p)}$ be defined as follows:

$$a_n^{(p)} = \frac{r(r+1)\cdots(r+p-n-1)}{(p-n)!} \quad \textit{when} \quad n < p;$$

$$a_n^{(p)} = 1 \quad \textit{when} \quad n = p; \qquad a_n^{(p)} = 0 \quad \textit{when} \quad n > p.$$

Then $S_p = S_p^{(r)}/D_p^{(r)}$ where $S_p^{(r)}$ and $D_p^{(r)}$ are given by (12). Condition (A) of the lemma is satisfied since

$$\lim_{p=\infty} \frac{\sum_{n=0}^{m} a_n^{(p)}}{\sum_{n=0}^{\infty} a_n^{(p)}} = \lim_{p=\infty} \frac{\sum_{n=0}^{m} a_n^{(p)}}{\frac{(r+1)(r+2)\cdots(r+p)}{p!}}$$

$$= \lim_{p=\infty} \left[\frac{r}{r+p} + \frac{rp}{(r+p)(r+p-1)} + \cdots \right.$$

$$\left. + \frac{rp(p-1)\cdots(p-m+1)}{(r+p)(r+p-1)\cdots(r+p-m)} \right] = 0.$$

Thus we have the desired result:

$$\lim_{p=\infty} S_p = \lim_{n=\infty} s_n = S,$$

provided the latter limit exists, *i. e.*, when (11) is convergent.

The consistency of (II) follows directly from that of (I) if we m the following established result: "If the limit s defined by (II) e given value of r then the limit s defined by (I) exists for the same valu conversely. Moreover, the two limits s are the same." In view of it appears that formulæ (I) and (II) are coextensive both in applic in the values of s which they associate with a given series (conver vergent). As the proof of the indicated result is lengthy, it will be omi

To show the consistency of (III), let P be taken as the continuo $p \geqq 0$ and let $a_n^{(p)} = p^n/n!$. Then

$$S_p = \frac{s_0 + s_1 p + s_2 \frac{p^2}{2!} + \cdots}{1 + p + \frac{p^2}{2!} + \cdots} = e^{-p} s(p).$$

Condition (A) is satisfied since

$$\lim_{p=\infty} e^{-p} \sum_{n=0}^{m} \frac{p^m}{m!} = 0. \tag{16}$$

Thus the lemma yields the desired result:

$$\lim_{p=\infty} e^{-p} s(p) = \lim_{\alpha=\infty} e^{-\alpha} s(\alpha) = \lim_{n=\infty} s_n = S. \tag{17}$$

In considering the consistency of (IV), we first note that when (vergent, $\lim_{n=\infty} u_n = 0$. Whence, if we apply the lemma of § 37 with s $a_n^{(p)} = p^n/n!$, noting also relation (16), we obtain

$$\lim_{p=\infty} e^{-p} u(p) = \lim_{\alpha=\infty} e^{-\alpha} u(\alpha) = \lim_{n=\infty} u_n = 0, \tag{18}$$

where $u(\alpha)$ has the meaning given in (14).

Now, from equation (17) together with $[e^{-\alpha} s(\alpha)]_{\alpha=0} = u_0$, we may

$$S - u_0 = \int_0^\infty \frac{d}{d\alpha}[e^{-\alpha} s(\alpha)] d\alpha.$$

But

$$\frac{d}{d\alpha}[e^{-\alpha} s(\alpha)] = e^{-\alpha}[s'(\alpha) - s(\alpha)],$$

where

$$\frac{d}{d\alpha} s(\alpha) = s'(\alpha) = s_1 + s_2 \alpha + s_3 \frac{\alpha^2}{2!} + \cdots.$$

Whence, if we note that

$$\frac{d}{d\alpha} u(\alpha) = u'(\alpha) = s'(\alpha) - s(\alpha) = u_1 + u_2 \alpha + u_3 \frac{\alpha^2}{2!} + \cdots,$$

[12] See Ford, *l. c.*, pp. 315–326. Also Schnee, *Math. Annalen*, Vol. 67 (1909), In view of this result we shall omit the detailed discussion of (II) throughout the pre all statements respecting it being identical with those obtained for (I).

we have

$$S - u_0 = \int_0^\infty e^{-\alpha} u'(\alpha) d\alpha. \tag{19}$$

Whence also, upon integrating by parts,

$$S - u_0 = \left[e^{-\alpha} \int_0^\alpha u'(\alpha) d\alpha \right]_0^\infty + \int_0^\infty e^{-\alpha} \left[\int_0^\alpha u'(\alpha) d\alpha \right] d\alpha$$
$$= \left[e^{-\alpha} \{u(\alpha) - u_0\} \right]_0^\infty + \int_0^\infty e^{-\alpha} \{u(\alpha) - u_0\} d\alpha.$$

Introducing (18) together with

$$u_0 = \int_0^\infty e^{-\alpha} u_0 d\alpha$$

we reach the desired relation:

$$\int_0^\infty e^{-\alpha} u(\alpha) d\alpha = S. \tag{20}$$

Definition (V) is at once seen to be consistent, for when (11) converges to S so also does the series

$$(u_0 + u_1 + \cdots + u_{p-1}) + (u_p + u_{p+1} + \cdots + u_{2p-1}) + (u_{2p} + u_{2p+1} + \cdots + u_{3p-1}) + \cdots,$$

and by applying (IV) to this series we obtain the desired result:

$$\int_0^\infty e^{-\alpha} u_k(\alpha) d\alpha = S.$$

Likewise, the consistency of (VI) may be shown by use of (20) for it is merely the application of this equation to the series

$$u_0 + 0 + 0 + \cdots + 0 + u_1 + 0 + 0 + \cdots + u_2 + 0 + \cdots,$$

wherein $p - 1$ zeros are inserted between each term and the preceding term in (11).

39. *The Boundary Value Condition.*—It is well known that two definitions of sum, both "consistent" (§ 37), do not necessarily give the *same* sum to a given divergent series. In other words, consistency alone is not an adequate principle upon which to base a scientific theory of summation because it does not insure uniqueness of sum.[13] A theory free from this objection may be

[13] See remarks in Preface. It would appear that many of the formulae for sum suggested within recent years have been obtained from considerations quite regardless of the question of uniqueness.

attained if (having demanded consistency) we confine the attentio series (11) for which the corresponding power series[14]

$$f(x) \equiv \sum_{n=0}^{\infty} u_n x^n \tag{21}$$

has a radius of convergence equal to 1 and then agree to retain those of sum for which

$$s = \lim_{x=1-0} f(x). \tag{B}$$

This procedure is in line with the historical genesis of the theory of su and allows the theory a well-defined usefulness in the study of analytic Indeed, if a general, self-consistent theory is to be formulated, it would it should contain (*B*), or an equivalent condition, though such a evidently tends to limit the immediate range of applicability of the t particular class of series (11) (cf. Preface).

Having assumed, then, that the series (11) is such that the power has a radius of convergence equal to 1, we shall undertake to deter present § those definitions of sum which satisfy (*B*). Definitions property we shall speak of as satisfying the *boundary value condition.*

We begin by showing that definition (I) satisfies (*B*), *i. e.*,

$$\lim_{x=1-0} \sum_{n=0}^{\infty} u_n x^n = \lim_{n=\infty} S_n^{(r)}/D_n^{(r)}, \tag{22}$$

whenever the latter limit exists. This may be done as follows by the lemma of § 37.

Let the s_n of the lemma be taken as $S_n^{(r)}/D_n^{(r)}$. Then place x so that as x ranges from a to 1 $(0 < a < x)$ the quantity p ranges fro to $+\infty$; also take $a_n^{(p)} = D_n^{(r)}(1 - 1/p)^n$. The expression S_p of then becomes

$$S_p = \frac{\sum_{n=0}^{\infty} S_n^{(r)} \left(1 - \frac{1}{p}\right)^n}{\sum_{n=0}^{\infty} D_n^{(r)} \left(1 - \frac{1}{p}\right)^n} = \frac{\sum_{n=0}^{\infty} S_n^{(r)} x^n}{\sum_{n=0}^{\infty} D_n^{(r)} x^n} = \frac{(1-x)^{-(r+1)} \sum_{n=0}^{\infty} u_n x^n}{(1-x)^{-(r+1)}} = \sum$$

[14] It is to be observed that this series is formed by supplying the *successive* pow (11) beginning with x^0, thus excluding, for example, the series (4) in connection wi of (1). This choice of $f(x)$, though arbitrary, is evidently the most natural and likely to result in a theory of summability having useful supplemental relations to t value problem.

[15] Some sum-formulae, such as (IV), § 36, not only satisfy (*B*) when applied for which (21) has a radius of convergence equal to 1, but they have the further they preserve a meaning in certain regions in which $|x| > 1$ and in these region analytical continuation of (21) (cf. § 44).

so that

$$\lim_{p=\infty} S_p = \lim_{x=1-0} \sum_{n=0}^{\infty} u_n x^n.$$

Let us now confine ourselves, as may be done without loss of generality, to values of p pertaining to the sequence $P \equiv 1, 2, 3, \cdots$. Condition (A) of the lemma is now satisfied, since

$$\lim_{p=\infty}\left[\sum_{n=0}^{m} D_n^{(r)}\left(1-\frac{1}{p}\right)^n \Big/ \sum_{n=0}^{\infty} D_n^{(r)}\left(1-\frac{1}{p}\right)^n\right]$$

$$= \lim_{p=\infty} \frac{1+\dfrac{(r+1)}{1}\left(1-\dfrac{1}{p}\right)+\dfrac{(r+1)(r+2)}{2!}\left(1-\dfrac{1}{p}\right)^2+\cdots}{1+\dfrac{(r+1)}{1}\left(1-\dfrac{1}{p}\right)+\dfrac{(r+1)(r+2)}{2!}\left(1-\dfrac{1}{p}\right)^2+\cdots},$$

which expression is evidently equal to zero since the denominator has a meaning for all $p > 0$ but becomes infinite with p, while the numerator remains finite as $p = \infty$.

Applying the lemma, we may therefore write (22) as desired.

We turn next to definition (III) and shall show that (B) is again satisfied, *i. e.*,

$$\lim_{x=1-0} \sum_{n=0}^{\infty} u_n x^n = \lim_{\alpha=\infty} e^{-\alpha} s(\alpha), \tag{23}$$

whenever the latter expression has a meaning.

For this purpose we first note that for any series (11) (convergent or divergent) for which the second member of (22) exists we have in the notation of § 36

$$\frac{d}{d\alpha}[e^{-\alpha}s(\alpha)] = e^{-\alpha}[s'(\alpha) - s(\alpha)] = e^{-\alpha}u'(\alpha); \qquad [e^{-\alpha}s(\alpha)]_{\alpha=0} = u_0 \tag{24}$$

and hence

$$\lim_{\alpha=\infty} e^{-\alpha}s(\alpha) = u_0 + \int_0^{\infty} \frac{d}{d\alpha}[e^{-\alpha}s(\alpha)]d\alpha = u_0 + \int_0^{\infty} e^{-\alpha}u'(\alpha)d\alpha. \tag{25}$$

Conversely, it appears from the same relations (24) that for any series (11) for which the last member of (25) exists, the second member of (23) exists also and we have relation (25).

This premised, let us return to the series (21). Since this series is convergent when $|x| < 1$ it follows from the consistency of definition (III) that when $0 < x < 1$

$$\sum_{n=0}^{\infty} u_n x^n = \lim_{\alpha=\infty} e^{-\alpha} s_x(\alpha),$$

where $s_x(\alpha)$ represents the function $s(\alpha)$ corresponding to the series (21). Whence, upon applying (25), we have also

$$\sum_{n=0}^{\infty} u_n x^n = u_0 + \int_0^{\infty} e^{-\alpha} u'(\alpha x) d\alpha; \qquad u'(\alpha x) = \frac{\partial}{\partial \alpha} u(\alpha x). \tag{26}$$

Assuming for the moment that the integral here appearing converges for all values of x in the interval $a < x < 1$; $a > 0$ we now have, usi

$$\lim_{x=1-0} \sum_{n=0}^{\infty} u_n x^n = u_0 + \int_0^{\infty} e^{-\alpha} u'(\alpha) d\alpha = \lim_{\alpha=\infty} e^{-\alpha} s(\alpha),$$

thus reaching the desired relation (23).

That the integral in (26) converges uniformly for values of x in t $a < x < 1$ may be established as follows: Place $\alpha x = y$ and subsequen x by $1/(1 + \theta)$. The integral under consideration thus takes the forn

$$(1 + \theta) \int_0^{\infty} e^{-\theta y} e^{-y} u'(y) dy, \tag{27}$$

so that it now suffices to show that (27) converges uniformly for all in the interval $0 < \theta < b$; $b = (1 - a)/a$.

Now, the integral

$$\int_0^{\infty} e^{-y} u'(y) dy \tag{28}$$

converges, as appears from (25), when we make use of our hypothes second member of (23) exists. Moreover, the expression $e^{-\theta y}$ is p steadily decreasing as y increases and it becomes equal to 1 for all when $y = 0$. We have therefore but to apply Abel's test[16] for t convergence of definite integrals to reach the desired result concernin

We proceed to show that definition (IV) also satisfies condition (I

$$\lim_{x=1-0} \sum_{n=0}^{\infty} u_n x^n = \int_0^{\infty} e^{-\alpha} u(\alpha) d\alpha, \tag{29}$$

whenever the latter expression has a meaning.

From the consistency of (IV) we have in the first place

$$\lim_{x=1-0} \sum_{n=0}^{\infty} u_n x^n = \lim_{x=1-0} \int_0^{\infty} e^{-\alpha} u(\alpha x) d\alpha,$$

so that it suffices for our purpose to prove that

$$\lim_{x=1-0} \int_0^{\infty} e^{-\alpha} u(\alpha x) d\alpha = \int_0^{\infty} e^{-\alpha} u(\alpha) d\alpha. \tag{30}$$

[16] Cf. Bromwich, *l. c.*, § 171 (2).

Placing $\alpha x = y$ and subsequently replacing x by $1/(1+\theta)$,[17] the integral in the first member of (30) takes the form

$$(1+\theta)\int_0^\infty e^{-\theta y}e^{-y}u(y)dy$$

and we may now show by applying Abel's test, as in the discussion of (III), that this integral converges uniformly for all values of θ in the interval $0 < \theta < b$; $b > 0$, with which the proof of (29) becomes complete.

Definition (V) does not in general satisfy condition (B), as appears from an example. Thus, let the series (21) be

$$\frac{1}{1+x} = \sum_{n=0}^\infty (-1)^n x^n$$

and take $p = 2$. Then $u_p(\alpha) \equiv 0$ and hence s, as given by (V), is equal to zero. But,

$$\lim_{x=1-0}\sum_{n=0}^\infty (-1)^n x^n = \frac{1}{1+1} = \tfrac{1}{2}.$$

That definition (VI) satisfies condition (B) may be readily inferred from reasoning similar to that followed in connection with (IV). Thus, from the consistency of the definition we have

$$\lim_{x=1-0}\sum_{n=0}^\infty u_n x^n = \lim_{x=1-0}\int_0^\infty e^{-\alpha}U_{p,x}(\alpha)d\alpha, \tag{31}$$

where

$$U_{p,x}(\alpha) = \sum_{n=0}^\infty \frac{u_n\alpha^{np}x^n}{(np)!}.$$

Upon placing $x = z^p$ the second member of (31) takes the form

$$\lim_{z=1-0}\int_0^\infty e^{-\alpha}U_p(\alpha z)d\alpha, \tag{32}$$

where U_p is defined by (15). Now place $\alpha z = y$ and subsequently replace z by $1/(1+\theta)$. Expression (32) then takes the form

$$\lim_{\theta=0}(1+\theta)\int_0^\infty e^{-y}e^{-\theta y}U_p(y)dy. \tag{33}$$

Since the integral

$$\int_0^\infty e^{-y}U_p(y)dy$$

has a meaning by hypothesis, we may show by means of Abel's test, as in connection with (IV), that the integral in (33) is uniformly convergent for all values of θ in the interval $0 < \theta < b$, with which the proof is at once completed.

[17] Cf. BROMWICH, *l. c.*, p. 121.

40. *Fundamental Operations.*—Besides being consistent and satisf boundary value condition (B),[18] it is evidently desirable that the sum to a numerical divergent series (11) shall, at least so far as possible, b which the usual operations applicable to convergent series are preserv operations of this type which we shall consider are the following:

(C) If s represents the sum of the divergent series (11) by a given d then the series

$$\sum_{n=k}^{\infty} u_n; \qquad k = \textit{positive integer} \tag{34}$$

shall have a sum $s^{(k)}$ by the same definition such that

$$s^{(k)} = s - (u_0 + u_1 + \cdots + u_{k-1}). \tag{35}$$

Conversely, if the series (34) has a sum $s^{(k)}$ by a given definition then (11) shall likewise have a sum s by the same definition and relation exist.

(D) If with a given definition of sum, the two divergent series:

$$\sum_{n=0}^{\infty} u_n, \qquad \sum_{n=0}^{\infty} v_n \tag{36}$$

have respectively the sums s_1, s_2, then the series

$$\sum_{n=0}^{\infty} (u_n \pm v_n)$$

shall possess by the same definition the sum $s_1 \pm s_2$.

(E) With the hypotheses stated in (D) the series

$$\sum_{n=0}^{\infty} w_n, \tag{37}$$

where

$$w_n = u_0 v_n + u_1 v_{n-1} + \cdots + u_{n-1} v_1 + u_n v_0$$

shall have the sum $s_1 s_2$ (at least after certain additional conditions h placed upon u_n and v_n analogous to those imposed when two *convergent* multiplied together).

We begin by showing that definition (I) satisfies condition (C). I evidently suffices to suppose $k = 1$, since a repetition of the reason from this to the general result (35).

[18] For reasons stated at the beginning of § 39 we shall continue throughout the r regard the given series (11) as belonging to the class for which the corresponding p (21) has a radius of convergence equal to 1. This hypothesis, however, plays no deductions about to be made.

Placing

$$(38)\qquad \begin{cases} s_n = u_0 + u_1 + \cdots + u_n, \\ \sigma_n = u_1 + u_2 + \cdots + u_{n+1}, \end{cases}$$

$${}_1S_n{}^{(r)} = s_n + rs_{n-1} + \frac{r(r+1)}{2!}s_{n-2} + \cdots + \frac{r(r+1)\cdots(r+n-1)}{n!}s_0,$$

$${}_2S_n{}^{(r)} = \sigma_n + r\sigma_{n-1} + \frac{r(r+1)}{2!}\sigma_{n-2} + \cdots + \frac{r(r+1)\cdots(r+n-1)}{n!}\sigma_0,$$

$$D_n{}^{(r)} = \frac{(r+1)(r+2)\cdots(r+n)}{n!},$$

we are to show, then, that if the limit

$$s_1 = \lim_{n=\infty} {}_1S_n{}^{(r)}/D_n{}^{(r)}$$

exists so also does the limit

$$s_2 = \lim_{n=\infty} {}_2S_n{}^{(r)}/D_n{}^{(r)}$$

and that $s_1 = u_0 + s_2$, with the corresponding converse statement.

Since $s_{n+1} = u_0 + \sigma_n$ we have

$${}_1S_{n+1}{}^{(r)} = cu_0 + \sigma_n + r\sigma_{n-1} + \frac{r(r+1)}{2!}\sigma_{n-2} + \cdots + \frac{r(r+1)\cdots(r+n-1)}{n!}\sigma_0$$

where

$$c = 1 + r + \frac{r(r+1)}{2!} + \cdots + \frac{r(r+1)\cdots(r+n-1)}{n!} = D_n{}^{(r)}.$$

Whence,

$$\frac{{}_1S_{n+1}{}^{(r)}}{D_n{}^{(r)}} = \frac{D_{n+1}{}^{(r)}}{D_n{}^{(r)}}\,\frac{{}_1S_{n+1}{}^{(r)}}{D_{n+1}{}^{(r)}} = u_0 + \frac{{}_2S_n{}^{(r)}}{D_n{}^{(r)}}.$$

The desired result (both direct and converse) now follows upon noting that

$$\lim_{n=\infty}\frac{D_{n+1}{}^{(r)}}{D_n{}^{(r)}} = 1.$$

As regards definition (III), it appears from an example that this does not always satisfy (C). Thus, consider the special series (11) for which $u_0, u_1, u_2, \cdots$ are so determined that

$$\sin(e^\alpha) = u_0 + (u_0 + u_1)\alpha + (u_0 + u_1 + u_2)\frac{\alpha^2}{2!} + \cdots.$$

For this series we have

$$s = \lim_{\alpha=\infty} e^{-\alpha}\sin(e^\alpha) = 0; \qquad s^{(1)} = \lim_{\alpha=\infty} e^{-\alpha}\left[\frac{d}{d\alpha}\sin(e^\alpha) - u_0e^\alpha\right] = \lim_{\alpha=\infty}\cos(e^\alpha) - u_0$$

so that although s exists, the same is not true of $s^{(1)}$.

In this connection we may, however, establish the following noteworthy result:

"If the series

(39) $$\sum_{n=p}^{\infty} u_n; \qquad p = 0, 1, 2, 3, \cdots, k$$

are each summable by (III) to the respective values s, $s^{(1)}$, $s^{(2)}$, $\cdots$, $s^{(k)}$, then relation (35) is satisfied."

In fact, with s_n and σ_n defined as in (38) and with

$$s(\alpha) = s_0 + s_1\alpha + s_2\frac{\alpha^2}{2!} + \cdots,$$

$$\sigma(\alpha) = \sigma_0 + \sigma_1\alpha + \sigma_2\frac{\alpha^2}{2!} + \cdots,$$

we have, since $s_{n+1} = u_0 + \sigma_n$,

$$s(\alpha) = u_0 + (u_0 + \sigma_0)\alpha + (u_0 + \sigma_1)\frac{\alpha^2}{2!} + \cdots = u_0e^{\alpha} + \left[\sigma_0\alpha + \sigma_1\frac{\alpha^2}{2!} + \cdots\right].$$

Or, since $\sigma_n = \sigma_{n+1} - u_{n+2}$,

$$s(\alpha) = u_0e^{\alpha} + \left[(\sigma_1 - u_2)\alpha + (\sigma_2 - u_3)\frac{\alpha^2}{2} + \cdots\right].$$

Whence also

$$e^{-\alpha}s(\alpha) = u_0 + e^{-\alpha}\sigma(\alpha) - e^{-\alpha}u'(\alpha),$$

where $u(\alpha)$ is determined from (14). It therefore remains but to show that $\lim e^{-\alpha}u'(\alpha) = 0$, in order to prove the indicated statement for the case in which $k = 1$.

Now, having assumed that both $\lim e^{-\alpha}s(\alpha)$ and $\lim e^{-\alpha}\sigma(\alpha)$ exist, it follows from the last equation that $\lim e^{-\alpha}u'(\alpha)$ exists also. Moreover, we have (cf. (25))

(40) $$\lim_{\alpha=\infty} e^{-\alpha}s(\alpha) = u_0 + \int_0^{\infty} e^{-\alpha}u'(\alpha)d\alpha$$

so that the only possible value of $\lim e^{-\alpha}u'(\alpha)$ is zero.[19]

Repetition of the reasoning now leads to the more general result as stated above.

Turning to definition (IV), it again appears that a series (11) which is summable by this definition may not satisfy (C), but that the following result may be established:[20]

"If the series (39) are summable by (IV) to the respective values s, $s^{(1)}$, $s^{(2)}$, $\cdots$, $s^{(k)}$, then relation (35) is satisfied."

[19] It may be observed that the existence of the integral in (40) does not suffice to establish the equation $\lim e^{-\alpha}u'(\alpha) = 0$ (cf. Bromwich, *l. c.*, p. 278).

[20] Cf. Hardy, "Researches in the Theory of Divergent Series, etc.," *Quarterly Journ. of Math.*, Vol. 35 (1904), p. 30.

In order to see the truth of this statement for the case in which $k = 1$ we first note that by an integration by parts we obtain

$$(41)\quad \begin{aligned} s = \int_0^\infty e^{-\alpha} u(\alpha) d\alpha = \Big[- e^{-\alpha} u(\alpha) \Big]_{\alpha=0}^{\alpha=\infty} \\ + \int_0^\infty e^{-\alpha} u'(\alpha) d\alpha = \Big[e^{-\alpha} u(\alpha) \Big]_{\alpha=0}^{\alpha=\infty} + s^{(1)}. \end{aligned}$$

From this relation combined with the assumed existence of s and $s^{(1)}$ it follows that $\lim e^{-\alpha} u(\alpha) = 0$ so that we have as desired $s^{(1)} = s - u_0$. In order to prove the more general case we have evidently but to repeat the same reasoning k times.

Definition (V) does not always satisfy condition (C) since, as we have just shown, it does not do so for the special case in which $p = 1$. Likewise, the same is true of definition (VI) (which reduces to (IV) when $p = 1$), but we here have an alternative result similar to that indicated above.

We turn then to condition (D). This is evidently satisfied by two series summable by any one of the definitions of § 36 and, therefore, needs no further comment.

As regards condition (E), it is obviously necessary to impose further conditions than that of the mere summability of the two series (36) in order that (E) be satisfied, at least in general, since even in the case of two *convergent* series such supplementary conditions are required. We here have, however, the following noteworthy result of Cesàro relative to series (36) summable by (I):

"The product series (37) of two series (36) whose degrees of indeterminacy (§ 36) are respectively r and s is summable and has a degree of indeterminacy no greater than $r + s + 1$."[21]

Conditions under which condition (E) will be satisfied by definition (IV) will be considered in § 44.

41. *Summary of Results.*—The principal results of §§ 35–40 may be summarized into the following statement:

Let

$$(42)\qquad \sum_{n=0}^{\infty} u_n$$

be any divergent series such that the corresponding power series

$$\sum_{n=0}^{\infty} u_n x^n$$

[21] We omit the proof of this well-known result. The same may be supplied from BROMWICH, *l. c.*, § 125. For CESÀRO's original proof, see *Bulletin des Sciences Math.*, Vol. 14 (1890), pp. 118, etc.

has a radius of convergence equal to 1. *Also, let* (I), (II), (III), (IV), (V) *and* (VI) *represent the six definitions for sum indicated in* § 36.

If, then, we represent by (A) *the condition of consistency* (§ 37), *by* (B) *the boundary value condition* (§ 39) *and by* (C), (D) *and* (E) *the conditions of* § 40 *carried over from the theory of convergent series, the relation of the various definitions to these conditions appears in the following table wherein the* * *when placed in any square indicates that the corresponding definition and condition are compatible:*

	I	II	III	IV	V	VI
A	*	*	*	*	*	*
B	*	*	*	*		*
C	*	*				
D	*	*	*	*	*	*
E						

FIG. 5

Moreover, the squares corresponding to (III, C), (IV, C) *and* (VI, C) *may also receive the* * *provided one substitutes for* (C) *the following slightly more restrictive condition:*

(C)′ *If the series*

$$\sum_{n=p}^{\infty} u_n; \qquad p = 0, 1, 2, 3, \cdots, k$$

are each summable in accordance with a given definition of sum to the respective values

$$s, \ s^{(1)}, \ s^{(2)}, \ s^{(3)}, \ \cdots, \ s^{(k)}$$

then

$$s^{(k)} = s - (u_0 + u_1 + \cdots + u_{k-1}).$$

42. *Absolutely Summable Series.*—A noteworthy class of divergent series (11) for which conditions (A), (B), (C), (D), (E), of § 41 are *all* satisfied when we adopt the definition (IV) of sum, has been pointed out by BOREL and made the object of especial study throughout his investigations.[22] Such series are called *absolutely summable* and are defined from the fact that not only the integral

$$s = \int_0^{\infty} e^{-\alpha} u(\alpha) d\alpha \tag{43}$$

is supposed to exist, but also each of the integrals

$$\int_0^{\infty} e^{-\alpha} |u^{(p)}(\alpha)| d\alpha; \qquad p = 0, 1, 2, 3, \cdots,$$

[22] Cf. "Leçons," Chapter III.

wherein $u^{(p)}(\alpha)$ denotes the pth derivative of the (integral) function $u(\alpha)$ (cf. (14)).

Absolutely summable series, as thus defined, being but special series summable by definition (IV), at once satisfy conditions (A), (B) and (D), as shown in earlier §§. It therefore remains but to consider such series with reference to conditions (C) and (E).

Now, if the series (11) is absolutely summable, it follows from definition that both s and $s^{(k)}$ exist. Whence, by the results obtained in § 40, we have relation (35). In order to complete the proof that (C) is satisfied, we must now show that if the series (34) is absolutely summable, so also is (11) and that with s and $s^{(k)}$ defined as before, relation (35) exists. For this let us first consider the case in which $k = 1$.

Place[23]

$$\varphi(x) = \int_0^x |u'(t)|\,dt \geqq \left|\int_0^x u'(t)dt\right|.$$

We thus have

$$\varphi(x) \geqq |u(x) - u_0|$$

and hence

$$|u(x)| \leqq \varphi(x) + |u_0|,$$

so that the integral

$$\int_0^\infty e^{-x}|u(x)|\,dx$$

must converge whenever the same is true of the integral

(44) $$\int_0^\infty e^{-x}\varphi(x)dx.$$

Now, by identity

$$\int_0^X e^{-x}\varphi(x)dx = -\,e^{-X}\varphi(X) + \int_0^X e^{-x}\varphi'(x)dx$$

and consequently, because $\varphi(X)$ and $\varphi'(x)$ are both positive,

$$\int_0^X e^{-x}\varphi(x)dx < \int_0^X e^{-x}\varphi'(x)dx < \int_0^\infty e^{-x}\varphi'(x)dx.$$

Thus the integrals (40) and (41) exist. Upon again applying the results obtained in § 40, the desired conclusion now follows for the case in which $k = 1$.

A repetition of the reasoning evidently leads to the more general result.

We proceed, then, to show that absolutely summable series satisfy condition (E).[24]

[23] Cf. Bromwich, *l. c.*, § 106.

[24] The proof which follows is essentially that given by Bromwich (*l. c.*, § 106).

In the first place, we may write (see definitions of s_1 and s_2 in (36), and note (43))

$$s_1s_2 = \lim_{\lambda=+\infty} \iint e^{-(x+y)}u(x)v(y)\,dxdy,$$

in which the double integral appearing in the second member is understood to be extended over the square $OABC$ of side λ situated as in the following figure:

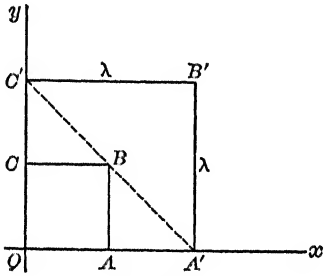

FIG. 6

In fact, we have

$$\lim_{\lambda=\infty} \iint e^{-(x+y)}u(x)v(y)\,dxdy = \lim_{\lambda=\infty} \int_0^\lambda \int_0^\lambda e^{-(x+y)}u(x)v(y)\,dxdy$$

$$= \lim_{\lambda=\infty} \left[\int_0^\lambda e^{-x}u(x)\,dx \int_0^\lambda e^{-y}v(y)\,dy\right] = \int_0^\infty e^{-x}u(x)\,dx \int_0^\infty e^{-y}v(y)\,dy = s_1s_2.$$

Now, in case $u(x)$ and $v(y)$ are always positive the indicated double integral when extended over the triangle $OA'C'$ has a value lying between the corresponding integrals taken over the squares $OABC$, $OA'B'C'$, and since the latter each approach the limit s_1s_2 as $\lambda = \infty$, the integral over the same triangle will also approach the limit s_1s_2. On the other hand, if $u(x)$ and $v(y)$ are not always positive, the absolute value of the difference between the integrals over $OABC$ and $OA'B'C'$ may be made arbitrarily small by taking λ sufficiently large, as we shall show presently, thus again rendering the integral over the triangle $OA'C'$ equal in the limit to s_1s_2.

In order to show this, let us represent by $I(S)$ the integral in question when extended over *any* given area S. Also let $G(S)$ be the corresponding integral when the absolute value of the integrand is used. We then have

$$(45) \quad \begin{aligned} |I(OABC) - I(OA'C')| &= |I(CBC') + I(AA'B)| \\ &\leqq |I(CBC')| + |I(AA'B)| < G(ABCC'B'A'A). \end{aligned}$$

Since $ABCC'B'A'A = OA'B'C' - OABC$ and since the integrand of $G(S)$ is always positive, the last member of (45) may be written in the form

$$(46) \quad \int_0^{2\lambda} e^{-x}|u(x)|\,dx \int_0^{2\lambda} e^{-y}|v(y)|\,dy - \int_0^{\lambda} e^{-x}|u(x)|\,dx \int_0^{\lambda} e^{-y}|v(y)|\,dy.$$

Moreover, since the series (36) are by hypothesis *absolutely* summable, each of the iterated integrals in (46) approaches the same limit when $\lambda = \infty$, so that the expression (46) itself approaches the limit zero.

We may therefore in all cases write

$$s_1 s_2 = \lim_{\lambda=\infty} \iint e^{-(x+y)} u(x) v(y) dx dy, \tag{47}$$

where the integration is performed over the right triangle $OA'C'$, the length of whose side is 2λ.

This result being premised, let us now introduce into the second member of (47) the new variables ξ, η defined as follows: $x + y = \xi$, $y = \xi\eta$ or $x = \xi(1 - \eta)$, $y = \xi\eta$.

We then have[25]

$$dydx = \begin{vmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\[2ex] \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{vmatrix} d\xi d\eta = \xi d\xi d\eta,$$

so that the integral in question becomes

$$\int_0^{2\lambda} e^{-\xi} \xi d\xi \int_0^1 u[\xi(1 - \eta)] v(\xi\eta) d\eta. \tag{48}$$

Concerning the limits of integration here, we wish to integrate over that area in the ξ, η plane which corresponds to the area of the triangle $OA'C'$ in the x, y plane. Now, the three sides of the triangle are respectively $x = 0$, $y = 0$ and $x + y = 2\lambda$, and our first problem is to determine what these bounding lines become in the ξ, η plane, it being understood as indicated above, that the equations of transformation are $x = \xi(1 - \eta)$, $y = \xi\eta$. Evidently, corresponding to $x = 0$ we have the two lines $\xi = 0$, $\eta = 1$, while corresponding to $y = 0$, we have the two lines $\xi = 0$, $\eta = 0$, and corresponding to $x + y = 2\lambda$ we have the one line $\xi = 2\lambda$. The area bounded by these four lines is that of the rectangle whose vertices (in the ξ, η plane) are $(0, 0)$, $(2\lambda, 0)$, $(2\lambda, 1)$, $(0, 1)$. Whence, the limits of integration are as indicated in (48).

The series for $u(x)$ and $v(y)$, being power series are *absolutely* convergent. Hence, by the rule for the multiplication of two such series, it appears that the expression $u[\xi(1 - \eta)]v(\xi\eta)$ may be expanded into a series whose nth term is

$$\xi^n \sum_{r=0}^{n} \frac{u_r v_{n-r}}{r!(n - r)!} \eta^{n-r} (1 - \eta)^r. \tag{49}$$

Moreover, this series will be uniformly convergent as regards η throughout the

[25] See, for example, Goursat, "Cours d'Analyse," Vol. I (1902), p. 298

interval $0 < \eta < 1$ since for all such η values the term (49) is less in absolute value than

$$\xi^n \sum_{r=0}^{n} \frac{|u_r|\,|v_{n-r}|}{r!(n-r)!}$$

and this expression is the nth term of the (convergent) product series obtained by multiplying together the (convergent) series

$$\sum_{n=0}^{\infty} \frac{|u_n|}{n!} \xi^n, \qquad \sum_{n=0}^{\infty} \frac{|v_n|}{n!} \xi^n.$$

The integration with respect to η in (48) may therefore be performed term by term upon the series whose nth term is (49), thus giving

$$\int_0^1 u[\xi(1-\eta)]v(\xi\eta)d\eta = \sum_{n=0}^{\infty} \xi^n \sum_{r=0}^{n} \frac{u_r v_{n-r}}{r!(n-r)!} \int_0^1 \eta^{n-r}(1-\eta)^r d\eta.$$

But

$$\int_0^1 \eta^{n-r}(1-\eta)^r d\eta = \frac{r!(n-r)!}{(n+1)!}.$$

Thus we have

$$\int_0^1 u[\xi(1-\eta)]v(\xi\eta)d\eta = \sum_{n=0}^{\infty} w_n \frac{\xi^n}{(n+1)!},$$

where w_n has the meaning used in condition (E) (§ 40).

The integral (48) thus becomes

$$\int_0^{2\lambda} e^{-\xi} W(\xi) d\xi,$$

where

$$W(\xi) = \sum_{n=0}^{\infty} w_n \frac{\xi^{n+1}}{(n+1)!}$$

and accordingly we have the equation

$$s_1 s_2 = \int_0^{\infty} e^{-\xi} W(\xi) d\xi. \tag{50}$$

The second member of this equation is seen to be the sum of the series

$$0 + w_0 + w_1 + \cdots, \tag{51}$$

so that our final result will now follow as soon as we show that under the existing hypotheses the series

$$w_0 + w_1 + w_2 + \cdots \tag{52}$$

is summable by definition (IV).

We may in fact show that the series (52) is *absolutely* summable. Moreover, since we have shown that absolutely summable series satisfy condition (C), it will here suffice to show that the series (51) is absolutely summable—*i. e.*, that the integrals

$$\int_0^\infty e^{-\xi}|W^{(k)}(\xi)|d\xi; \qquad k = 0, 1, 2, 3, \cdots$$

converge. The proof of this presents no difficulties and will therefore be omitted.[26]

43. *Uniform Summability.*—Following analogy with uniformly convergent series, HARDY[27] has proposed the following definition of uniform summability for divergent series, basing the same on the form (IV) (§ 36) of definition of sum:

Definition I. If (instead of the series of constant terms (11)) we have the series (convergent or divergent)

$$\sum_{n=0}^{\infty} u_n(\alpha),$$

in which each term $u_n(\alpha)$ is a function of the (real) variable α, this series is *uniformly summable* throughout the interval $\beta < \alpha < \gamma$ if for these values of α the integral

$$\overset{\infty}{\underset{0}{\mathrm{S}}}\, u_n(\alpha) \equiv \int_0^\infty e^{-x}u(x, \alpha)dx$$

converges *uniformly*, wherein

$$u(x, \alpha) = \sum_{n=0}^{\infty} u_n(\alpha)\frac{x^n}{n!}.$$

Upon the basis of this definition the following theorems analogous to those encountered in the study of uniformly convergent series may be established:[28]

THEOREM I. "*If all the terms $u_n(\alpha)$ are continuous functions of α and*

$$\overset{\infty}{\underset{n=0}{\mathrm{S}}}\, u_n(\alpha)$$

is uniformly summable, and

$$\sum_{n=0}^{\infty} u_n(\alpha)\frac{x^n}{n!}$$

uniformly convergent for any finite value of x, in an interval (β, γ), the sum of the first series is a continuous function of α throughout the interval."

THEOREM II. "*If*

$$\overset{\infty}{\underset{n=0}{\mathrm{S}}}\, u_n'(\alpha)$$

is uniformly summable in ($\alpha_0 - \xi$, $\alpha_0 + \xi$) and

[26] Cf. BROMWICH, *l. c.*, pp. 282–283.

[27] See *Transactions Cambridge Philos. Soc.*, Vol. 19 (1904), p. 301.

[28] Cf. HARDY, *l. c.*

$$\sum_{n=0}^{\infty} u_n'(\alpha)\frac{x^n}{n!}$$

uniformly convergent for any finite value of x, the series

$$\mathop{\mathrm{S}}_{n=0}^{\infty} u_n(\alpha)$$

may be differentiated term by term for $\alpha = \alpha_0$."

THEOREM III. "*If*

$$\mathop{\mathrm{S}}_{n=0}^{\infty} u_n(\alpha) \tag{53}$$

is uniformly summable in (β, γ) and

$$\sum_{n=0}^{\infty} u_n(\alpha)\frac{x^n}{n!}$$

uniformly convergent throughout the domain $(0, X, \beta, \gamma)$ *however great be X, the series may be integrated term by term over (β, γ).*"

Extensions of Theorem III to cases in which (53) fails to be uniformly summable in the neighborhood of a finite number of isolated points within (β, γ) and to the case in which $\beta = \infty$ have also been obtained. It would appear, however, that with the indicated meaning for

$$\mathop{\mathrm{S}}_{n=0}^{\infty} u_n(\alpha),$$

Theorems I, II and III together with their generalizations relate in substance to the properties of definite integrals of a certain prescribed type rather than to the subject of infinite series, the latter appearing merely in the rôle of suggesting the type in question. For this reason the notion of "uniform summability," at least as formulated upon the basis of definition (IV) (§ 36), together with the resulting theorems appear somewhat artificial. This seems less true, however, in case definition (I) (or (II)) is adopted. Thus, confining ourselves for simplicity to the important case in which $r = 1$, we then have the following

Definition II.[29] A series (convergent or divergent)

$$\sum_{n=0}^{\infty} u_n(\alpha) \tag{54}$$

in which each term $u_n(\alpha)$ is a function of the (real) variable α, is uniformly summable throughout the interval $\beta < \alpha < \gamma$ if for these values of α the expression

$$\frac{s_0(\alpha) + s_1(\alpha) + \cdots + s_n(\alpha)}{n+1} \qquad \textit{where} \qquad s_n(\alpha) = u_0(\alpha) + u_1(\alpha) + \cdots + u_n(\alpha)$$

converges uniformly to a limit $U(\alpha)$.

[29] Cf., for example, C. N. MOORE, *Transactions American Math. Soc.*, Vol. 10 (1909), p. 400.

The theorems corresponding to I, II and III now become considerably more direct. Thus, corresponding to Theorem I we evidently have the following:

"If all the terms $u_n(\alpha)$ of the series (54) are continuous and the same series is uniformly summable throughout the interval (β, γ), then its sum $U(\alpha)$ is continuous throughout (β, γ)."

The corresponding forms for Theorems II and III can be at once supplied.

Supplementary Remarks and Theorems

44. From §§ 41–43 it may be concluded that of the six definitions of "sum" in § 36 those deserving of especial emphasis are (I) (Cesàro) or its equivalent (II) (Hölder) and (IV) (Borel). We now add certain noteworthy results respecting (I) and (IV), omitting proofs in cases where suitable references can be given.

1. *If a series (convergent or divergent) is summable by Cesàro's method for a given value of* r (cf. § 36), *it is summable by the same method for all larger (integral) values of* r.

In fact, with $S_n^{(r)}$ and $D_n^{(r)}$ defined as in (12), we have the identities

$$S_n^{(r+1)} = S_0^{(r)} + S_1^{(r)} + S_2^{(r)} + \cdots + S_n^{(r)},$$

$$D_n^{(r+1)} = D_0^{(r)} + D_1^{(r)} + D_2^{(r)} + \cdots + D_n^{(r)},$$

and since by hypothesis $\lim S_n^{(r)}/D_n^{(r)}$ exists, it follows from a well-known theorem due to Stolz[30] that $\lim S_n^{(r+1)}/D_n^{(r+1)}$ also exists and has the same value, provided however that as n increases $S_n^{(r)}$ eventually does not oscillate but is such that $\lim\limits_{n=\infty} S_n^{(r)} = \pm \infty$—a condition here fulfilled because by hypothesis $\lim S_n^{(r)}/D_n^{(r)}$ exists, while from the definition of $D_n^{(r)}$ we have at once $\lim D_n^{(r)} = +\infty$.

2. *A necessary condition that any series*

$$\sum_{n=0}^{\infty} u_n$$

be summable by Cesàro's method with a degree of indeterminacy r is that

$$\lim_{n=\infty} (u_n/n^r) = 0. \tag{55}$$ [31]

A noteworthy corollary of this result is as follows:

3. *Let*

$$\sum_{n=0}^{\infty} a_n x^n \tag{56}$$

[30] See *Math. Annalen*, Vol. 33 (1889), pp. 236–245.

[31] For a proof, see Bromwich, *l. c.*, § 127.

be any power series having a radius of convergence equal to 1. *Then the divergent series*

$$\sum_{n=0}^{\infty} a_n x_0{}^n,$$

wherein x_0 *represents any special value such that* $|x_0| > 1$ *cannot be summed by Cesàro's method.* Thus, in particular Cesàro's formula cannot serve to prolong analytically the power series (56) outside its circle of convergence.

In fact, placing $u_n = a_n x_0{}^n$ we have

$$\lim_{n=\infty} \frac{u_n}{u_{n-1}} = x_0$$

and hence

$$\frac{u_n}{u_{n-1}} = x_0 + \epsilon_n; \qquad \lim_{n=\infty} \epsilon_n = 0.$$

Whence

$$u_n = u_0(x_0 + \epsilon_1)(x_0 + \epsilon_2) \cdots (x_0 + \epsilon_n).$$

Now, having chosen an arbitrarily small positive quantity η, we have $|\epsilon_n| < \eta$ for all $n >$ a determinate value n_η, and hence

$$|x_0 + \epsilon_n| \geqq |x_0| - |\eta|; \qquad n > n_\eta.$$

Thus, as n increases indefinitely the expression u_n becomes infinite to as high an order as that of $(|x_0| - |\eta|)^n$. But for a sufficiently small choice of η we have $|x_0| - |\eta| > 1$, since by hypothesis $|x_0| > 1$. Thus (55) cannot be satisfied for any value of r.

In contrast to this result, we have the following important theorem arising when, instead of the definition (I) of sum, we adopt the definition (IV) of Borel.

4. *Let*

$$f(x) \equiv \sum_{n=0}^{\infty} a_n x^n$$

be any power series having a radius of convergence equal to 1. *If, then, the series*

$$\sum_{n=0}^{\infty} a_n$$

is summable by definition (IV) (§ 36) *so also is the series*

$$\sum_{n=0}^{\infty} a_n x_0{}^n$$

provided x_0 *lie within the polygon formed by tangents to the given circle at the points (assumed finite in number) upon the circumference at which* $f(x)$ *has singularities. Moreover,* $f(x)$ *may be extended analytically to all such points* x_0 *by means of the sum formula in question, i. e.,*

$$f(x_0) = \int_0^\infty e^{-\alpha} u(\alpha x_0) d\alpha$$

where

$$u(\alpha x_0) = \sum_{n=0}^{\infty} \frac{a_n (\alpha x_0)^n}{n!}.$$

The summability at x_0 will be absolute (§ 42) *and it will be uniform* (§ 43) *throughout any region situated wholly within the indicated polygon* (polygon of summability).[32]

5. *Absolutely convergent series are absolutely summable, but series that are merely convergent may not be absolutely summable.*[33]

6. *If but one of two series is absolutely summable while both are summable by definition* (IV) (*Borel's integral*) *to the respective limits s_1, s_2, then the product series* (cf. (37)) *is summable by the same definition to the value s_1, s_2, but not necessarily absolutely summable.*[34]

7. *If two series are summable by definition* (IV) (*Borel's integral*) *to the values s_1, s_2 respectively, then the product series* (cf. (37)) *whenever summable necessarily has the sum $s_1 s_2$.*[35]

8. *If the coefficients u_1, u_2, u_3, $\cdots$ of the divergent series* (11) *are such that the expressions*

$$(57)\qquad \begin{aligned} &E_0 = u_0, \qquad E_1 = u_0 + u_1, \\ &E_2 = u_0 + 2u_1 + u_2, \\ &E_3 = u_0 + 3u_1 + 3u_2 + u_4, \\ &\vdots \qquad\qquad \vdots \\ &E_n = u_0 + nu_1 + \frac{n(n-1)}{2!} u^3 + \cdots + nu_{n-1} + u_n \end{aligned}$$

all vanish after a certain point: $n = m$, then the series may be summed by definition (IV) (*Borel's integral*) *and the sum will be*

$$s = \frac{E_0}{2} + \frac{E_1}{2^2} + \frac{E_2}{2^3} + \cdots + \frac{E_m}{2^{m+1}}$$

—*i. e.*, the sum will be given by summing the series by Euler's well-known method for converting a slowly convergent series into a more rapidly converging one.[36]

[32] Proof of the various statements here made is readily supplied from the remarks of Bromwich, *l. c.*, § 113.

[33] Cf. Hardy, *Quarterly Journ. of Math.*, Vol. 35 (1904), pp. 25, 28.

[34] Cf. Hardy, *l. c.*, pp. 43–44.

[35] Cf. Hardy, *l. c.*, pp. 44–45

[36] Cf. Bromwich, *l. c.*, § 24.

This result evidently becomes of especial significance for all series (11) of the form

$$a_0 - a_1 + a_2 - a_3 + \cdots; \qquad a_m \textit{ positive}$$

for which the successive differences between the quantities a_0, a_1, a_2, $\cdots$ all eventually vanish—*e. g.*, the series

$$1 - 2 + 3 - 4 + 5 - \cdots,$$

wherein the quantities E_0, E_1, E_2, etc., become

$$E_0 = 1, \qquad E_1 = -1, \qquad E_2 = E_3 = \cdots = E_n = 0,$$

and hence $s = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

The proof of statement (8) may be readily supplied when we make use of the Lemma of § 37. Thus, in the notation there employed, let us take in the present instance

$$s_n = \frac{E_0}{2} + \frac{E_1}{2^2} + \frac{E_2}{2^3} + \cdots + \frac{E_n}{2^{n+1}}; \qquad a_n{}^{(p)} = \frac{(2p)^n}{n!}.$$

Then

$$l = \lim_{n=\infty} s_n = \frac{E_0}{2} + \frac{E_1}{2^2} + \cdots + \frac{E_m}{2^{m+1}}$$

and condition (A) of the lemma is at once seen to be satisfied (cf. (16)).

Application of the lemma thus gives

$$l = \lim_{p=\infty} e^{-2p} \sum_{n=0}^{\infty} \frac{u^{(d z)}}{n!} s_n = \lim_{\alpha=\infty} e^{-2\alpha} s(2\alpha) = \lim_{\alpha=\infty} e^{-\alpha} s(\alpha),$$

where $s(\alpha)$ is defined by (13). Moreover, this result may be written (cf. (25)) in the form

$$l = u_0 + \int_0^\infty e^{-\alpha} u'(\alpha)\, d\alpha,$$

where $u(\alpha)$ is defined by (14). If the integral here appearing be integrated by parts (cf. (41)) we thus obtain

$$l = -[e^{-\alpha} u(\alpha)]_{\alpha=\infty} + \int_0^\infty e^{-\alpha} u(\alpha)\, d\alpha.$$

In order to finish the proof it remains but to show that the first term here appearing in the second member is equal to zero.

Upon noting the meanings of E_0, E_1, E_2, $\cdots$, as given in (57), we obtain

$$e^{\alpha} u(\alpha) = e^{\alpha}\left(u_0 + u_1\alpha + u_2\frac{\alpha^2}{2!} + \cdots\right) = E_0 + E_1\alpha + E_2\frac{\alpha^2}{2!} + \cdots + E_m\frac{\alpha^m}{m!}$$

and hence

$$\lim_{\alpha=\infty} e^{-\alpha} u(\alpha) = \lim_{\alpha=\infty} e^{-2\alpha}\left[E_0 + E_1\alpha + \cdots + E_m\frac{\alpha^m}{m!}\right] = 0.$$

CHAPTER V

THE SUMMABILITY AND CONVERGENCE OF FOURIER SERIES AND ALLIED DEVELOPMENTS

45. In the present chapter it is proposed to derive the principal known results concerning the summability of Fourier series and other allied developments for functions of one real variable (developments in terms of Bessel functions, Legendre functions, etc.).[1] We shall take the word "sum" in the Hölder sense[2] (§ 36) according to which a given series (convergent or divergent)

$$\sum_{n=0}^{\infty} u_n \tag{1}$$

has its sum s defined by the equation

$$s = \lim_{n=\infty} s_n^{(r)}; \qquad r = \textit{fixed integer} \geqq 0, \tag{2}$$

where

$$\begin{aligned} s_n^{(0)} &= s_n = u_0 + u_1 + \cdots + u_n, \\ s_n^{(1)} &= \frac{1}{n+1}\left(s_0^{(0)} + s_1^{(0)} + \cdots + s_n^{(0)}\right), \\ &\;\vdots \\ s_n^{(r)} &= \frac{1}{n+1}\left(s_0^{(r-1)} + s_1^{(r-1)} + \cdots + s_n^{(r-1)}\right). \end{aligned}$$

Moreover, if the terms u_n are functions of the (real) variable x (as will now always be the case) when considered throughout an interval (a, b), the series (1) will be termed *uniformly* summable throughout (a, b) in accordance with the definition II of § 43—*i. e.*, provided that the limit (2) is approached uniformly for the same values of x.

In view of the fact that any discussion of the summability of Fourier series and other allied developments is intimately connected with the corresponding discussion of convergence, the latter being in fact but the case of summability in which $r = 0$ (cf. (2)), we shall as a matter of course elaborate both aspects of the subject.[3] No attempts will be made however to obtain theorems containing the minimum restrictions for a given function $f(x)$ in order that it be

[1] See explanatory remarks in the Preface.

[2] The results obtained will therefore (§ 38) be convertible at any point into those for summability in the Cesàro sense.

[3] Since all convergent series are summable but not conversely it is evident that more restrictive conditions upon u_n are in general necessary to insure convergence than summability.

developable in a summable (or convergent) series of any one type. The emphasis will be placed rather upon the attainment of a *general* theory of such a nature that the various more important special developments, including Fourier series, and the familiar developments in terms of Bessel functions and Legendre functions, may be studied as special applications of it, provided $f(x)$ satisfy any one of various slightly limiting conditions.[4] This general theory is elaborated in §§ 46–56 following which the applications just mentioned have been carried through (§§ 57–70).

The basis of the entire chapter is DINI's great work entitled "Serie di Fourier e altre rappresentazioni analitiche delle funzioni di una variabile reale" (Pisa, 1880) and due acknowledgment is here made to this source.

I

FOURIER SERIES

46. If $f(x)$ be a given function of the real variable x defined throughout the interval $(-\pi, \pi)$ the corresponding Fourier series is by definition

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{3}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx dx.$$

As regards the convergence and summability of this series, the following theorems are well known:

THEOREM I. *If $f(x)$ remains finite throughout the interval $(-\pi, \pi)$ with the possible exception of a finite number of points and is such that the integral*

$$\int_{-\pi}^{\pi} |f(x)|\, dx \tag{4}$$

exists, then the Fourier series (3) *will converge at any point x $(-\pi < x < \pi)$ in the arbitrarily small neighborhood of which $f(x)$ has limited total fluctuation, and the sum will be*

$$\frac{1}{2}[f(x-0) + f(x+0)].$$

Moreover, the convergence will be uniform to the limit $f(x)$ throughout any interval (a', b') inclosed within a second interval (a_1, b_1) such that

$$-\pi < a_1 < a' < b' < b_1 < \pi$$

[4] As regards *convergence*, including uniform convergence, general theorems of the nature here indicated together with applications have been given by HOBSON in a series of memoirs appearing in the *Transactions of the London Math. Society* (Vol. 6 (1908), pp. 349–395; Vol. 7, pp. 24–48; *ibid.*, pp. 338–388). Corresponding general studies for summability do not appear to have been thus far carried through, though numerous results have been obtained by special methods. For further remarks, see notes appended to the theorems of §§ 67 and 68.

provided that $f(x)$ *is continuous throughout* (a', b') *inclusive of the end points* $x = a'$, $x = b'$ *and has limited total fluctuation throughout* (a_1, b_1).[5]

THEOREM II. *If* $f(x)$ *remains finite throughout the interval* $(-\pi, \pi)$ *with the possible exception of a finite number of points and is such that the integral* (4) *exists, then the Fourier series* (3) *will be summable* ($r = 1$) *at any point* x $(-\pi < x < \pi)$ *at which the limits* $f(x-0)$, $f(x+0)$ *exist, and the sum will be*

$$\tfrac{1}{2}\,[f(x-0)+f(x+0)].$$

Moreover, the summability will be uniform to the limit $f(x)$ *throughout any interval* (a', b') *such that* $-\pi < a' < b' < \pi$ *provided that* $f(x)$ *is continuous throughout* (a', b') *inclusive of the end points.*[6]

THEOREM III. *If* $f(x)$ *when considered throughout the interval* $(-\pi, \pi)$ *satisfies the conditions mentioned in Theorem I and is such that in arbitrarily small neighborhoods at the right of the point* $x = -\pi$ *and at the left of the point* $x = \pi$ *it has limited total fluctuation, then the Fourier series* (3) *will converge when* $x = -\pi$ *or* $x = \pi$ *and in either case the sum will be*

$$\tfrac{1}{2}\,[f(\pi-0)+f(-\pi+0)].$$

THEOREM IV. *If* $f(x)$ *when considered throughout the interval* $(-\pi, \pi)$ *satisfies the conditions mentioned in Theorem I and is such that the limits* $f(\pi-0)$, $f(-\pi+0)$ *exist, then the Fourier series* (3) *will be summable when* $x = \pi$ *or* $x = -\pi$ *and in either case the sum will be*

$$\tfrac{1}{2}\,[f(\pi-0)+f(-\pi+0)].$$

It is our purpose here (having in mind the essential steps incident to the formation of a general theory for the study of this and other allied developments) to show in the first place that the proof of Theorem I may be made to depend upon the existence of the three following relations which themselves are independent of the function $f(x)$ and concern only the trigonometric expression

$$\varphi(n, t) = \frac{\sin\dfrac{2n+1}{2}t}{2\pi\sin\dfrac{t}{2}}, \tag{5}$$

n being limited to positive integral values.

(I) The integral

$$\int_0^t \varphi(n, t)\,dt$$

[5] Cf. HOBSON, "Theory of Functions," §§ 448, 451, 457, 459. Also, CHAPMAN, *Quarterly Journ. of Math.*, Vol. 43 (1911), p. 33.

[6] Cf. HOBSON, *l. c.*, § 469.

when considered for values of t in the interval $-2\pi + \epsilon \overline{\leqq} t \overline{\leqq} -\epsilon$, ϵ being an arbitrarily small positive constant, converges uniformly to the limit $-\frac{1}{2}$ when $n = \infty$; while the same integral when considered for values of t in the interval $\epsilon \leqq t \leqq 2\pi - \epsilon$ converges uniformly to the limit $\frac{1}{2}$ when $n = \infty$.[7]

(II) For a sufficiently small choice of the positive quantity ϵ we have

$$\left| \int_0^t \varphi(n, t)dt \right| < A; \qquad -\epsilon \leqq t \leqq \epsilon,$$

where A is a constant independent of both n and t.[8]

(III) For a sufficiently small choice of the positive quantity ϵ we have

$$|\varphi(n, t)| < B; \qquad -2\pi + \epsilon \leqq t \leqq -\epsilon, \qquad \epsilon \leqq t \leqq 2\pi - \epsilon,$$

where B is a constant independent of both n and t.

In order to prove that Theorem I depends, as stated above, upon the existence of these three relations, let us suppose at first that x has some special value $x = \alpha$ such that $-\pi < a' \leqq \alpha \leqq b' < \pi$, the quantities a', b' being regarded as fixed. With this value of α the $(n + 1)$st term of the series (3) takes the form

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)(\cos nx \cos n\alpha + \sin nx \sin n\alpha)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos n(x - \alpha)dx,$$

so that the sum of the first $(n + 1)$ terms becomes

$$s_n(\alpha) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \left\{ \frac{1}{2} + \sum_{n=1}^{n} \cos n(x - \alpha) \right\} dx.$$

Upon making use of the well-known relation

$$\frac{1}{2} + \sum_{n=1}^{n} \cos nx = \frac{\sin \dfrac{2n + 1}{2} x}{2 \sin \dfrac{x}{2}}$$

we thus have

$$s_n(\alpha) = \int_{-\pi}^{\pi} f(x)\varphi(n, x - \alpha)dx, \tag{6}$$

where $\varphi(n, x - \alpha)$ is to be determined by (5).

Whence also, having chosen an arbitrarily small positive quantity ϵ, we may write

$$\begin{aligned} s_n(\alpha) = &\int_{-\pi}^{\alpha-\epsilon} f(x)\varphi(n, x - \alpha)dx + \int_{\alpha+\epsilon}^{\pi} f(x)\varphi(n, x - \alpha)dx \\ &+ \int_{\alpha-\epsilon}^{\alpha} f(x)\varphi(n, x - \alpha)dx + \int_{\alpha}^{\alpha+\epsilon} f(x)\varphi(n, x - \alpha)dx. \end{aligned} \tag{7}$$

[7] For a proof of this statement, see Appendix, § 1.

[8] For a proof, see Appendix, § 2.

We may now show that the conditions placed upon $f(x)$ for the whole (closed) interval $(-\pi, \pi)$ (cf. Theorem I) when taken in conjunction with relations (I) and (III) suffice to make the limit approached by each of the first two integrals of (7) equal to zero when $n = \infty$. In fact, we shall show that this limit is approached *uniformly* by each of these two integrals when they are considered for values of α for which $a' \leqq \alpha \leqq b'$.

Considering, then, the first integral in the second member of (7), let us represent by $x_1, x_2, x_3, \cdots, x_q$ $(x_s > x_{s-1})$ the points (q in number) at which $f(x)$ becomes infinite in the (closed) interval $(-\pi, \pi)$, assuming at first for simplicity that $x_1 \neq -\pi$, $x_q \neq \pi$. Having chosen an arbitrarily small positive quantity ω, let us also suppose at first that the value $x = \alpha - \epsilon$ lies within one of the following intervals:

(8) $\qquad (-\pi, x_1 - \omega), \qquad (x_1 + \omega, x_2 - \omega), \qquad \cdots, \qquad (x_q + \omega, -\pi),$

i. e., let us assume that $x = \alpha - \epsilon$ is not one of the points at which $f(x)$ becomes infinite. We may then express the integral in question in the form

(9) $$\int_{-\pi}^{\alpha-\epsilon} f(x)\varphi(n, x - \alpha)dx = S + R,$$

where

$$S = \left(\int_{-\pi}^{x_1-\omega} + \int_{x_1+\omega}^{x_2-\omega} + \cdots + \int_{x_{g-1}+\omega}^{x_g-\omega} + \int_{x_g+\omega}^{\alpha-\epsilon}\right) f(x)\varphi(n, x - \alpha)dx, \qquad (g \leqq q)$$

and

$$R = \left(\int_{x_1-\omega}^{x_1+\omega} + \int_{x_2-\omega}^{x_2+\omega} + \cdots + \int_{x_g-\omega}^{x_g+\omega}\right) f(x)\varphi(n, x - \alpha)dx.$$

Now, introducing relation (III), we have

$$|R| < B\left(\int_{x_1-\omega}^{x_1+\omega} + \int_{x_2-\omega}^{x_2+\omega} + \cdots + \int_{x_g-\omega}^{x_g+\omega}\right) |f(x)|\,dx$$

and since the integral (4) is assumed to exist it thus follows that for a sufficiently small choice of ω we shall have $|R| < Bg\rho < Bq\rho$ where ρ is a preassigned arbitrarily small positive constant.

The value of ρ having been assigned and ω then determined in the manner just indicated, we turn to the expression S. In considering this it is first desirable to make the following observation.

Consider the set of intervals (8). Let us divide the first of these into p equal sub-intervals of length δ_1, the second into the same number p of equal sub-intervals $\delta_2, \cdots$, the $(q + 1)$st into the same number p of equal sub-intervals of length δ_{q+1}. Let $D_{1,s}$ be the fluctuation of $f(x)$ in the sth one of the intervals δ_1, let $D_{2,s}$ be the fluctuation of $f(x)$ in the sth one of the intervals $\delta_2, \cdots$, let $D_{q+1,s}$ be the fluctuation of $f(x)$ in the sth one of the intervals δ_{q+1}. Finally, let us form the sums

(10) $$\delta_1 \sum_{s=1}^{p} D_{1,\,s}, \qquad \delta_2 \sum_{s=1}^{p} D_{2,\,s}, \qquad \cdots, \qquad \delta_{q+1} \sum_{s=1}^{p} D_{q+1,\,s}.$$

Since $f(x)$ is integrable over each of the intervals (8), it follows that we can make our choice of the integer p so large that each of the sums (10) will be less in absolute value than the preassigned quantity ρ already mentioned. At the same time, p may be chosen so large that each of the integrals

(11) $$\int_{\delta_s} |f(x)|\,dx; \qquad s = 1, 2, 3, \cdots, q+1,$$

where the integration is performed over any one of the intervals δ_s, will likewise be less than ρ. In what follows the quantity p will be understood to be any special one determined according to the two conditions just indicated.

Returning to the expression S, let us now consider the first of the integrals of which it is constituted. Calling $x = \xi_{s-1}$, $x = \xi_s$ the values of x corresponding to the end points of the sth one of the intervals δ_1, we have

$$\int_{-\pi}^{x_1-\omega} f\cdot\varphi dx = \sum_{s=1}^{p}\int_{\xi_{s-1}}^{\xi_s} f\cdot\varphi dx; \qquad \begin{cases} f = f(x) \\ \varphi = \varphi(n, x-\alpha). \end{cases}$$

Now, introducing the constant B defined in (III), we may write

$$\int_{\xi_{s-1}}^{\xi_s} f\cdot\varphi dx = \int_{\xi_{s-1}}^{\xi_s} f\cdot(\varphi + B)dx - B\int_{\xi_{s-1}}^{\xi_s} fdx$$

and the function $\varphi + B$ will be positive for all values of x such that $\xi_{s-1} < x < \xi_s$ (n having any value which it may take). Hence, upon applying the first law of the mean for integrals, we have

$$\int_{\xi_{s-1}}^{\xi_s} f\cdot\varphi dx = f_s\int_{\xi_{s-1}}^{\xi_s}\varphi dx + Bf_s\int_{\xi_{s-1}}^{\xi_s} dx - Bf_s'\int_{\xi_{s-1}}^{\xi_s} dx,$$

where f_s and f_s' are certain values lying between the upper and lower limits of $f(x)$ when $\xi_{s-1} < x < \xi_s$.

Since $\xi_s - \xi_{s-1} = \delta_1$ we thus have

$$\int_{\xi_{s-1}}^{\xi_s} f\cdot\varphi dx = f_s\int_{\xi_{s-1}}^{\xi_s}\varphi dx + \theta_s B\delta_1 D_{1,\,s},$$

where θ_s is a quantity lying between -1 and $+1$ and where $D_{1,\,s}$ has the meaning already indicated.

Hence, recalling what has been said of the sums (10), we may write

(12) $$\int_{-\pi}^{x_1-\omega} f\cdot\varphi dx = \sum_{s=1}^{p} f_s\int_{\xi_{s-1}}^{\xi_s}\varphi dx + \theta B\rho; \qquad -1 < \theta < 1.$$

Now,

$$\int_{\xi_{s-1}}^{\xi_s} \varphi dx = \int_{\xi_{s-1}-\alpha}^{\xi_s-\alpha} \varphi(n, t)dt = \int_0^{\xi_s-\alpha} \varphi(n, t)dt - \int_0^{\xi_{s-1}-\alpha} \varphi(n, t)dt$$

and corresponding to a second arbitrarily small positive quantity σ we may, by virtue of relation (I), find a quantity n_σ *independent of* α such that

$$\left.\begin{aligned} \int_0^{\xi_s-\alpha} \varphi(n, t)dt &= -\tfrac{1}{2} + \Theta_1\sigma \\ \int_0^{\xi_{s-1}-\alpha} \varphi(n, t)dt &= -\tfrac{1}{2} + \Theta_2\sigma \end{aligned}\right. \qquad \begin{cases} n > n_\sigma, \\ -1 < \Theta_1 < 1, \quad -1 < \Theta_2 < 1, \\ a' < \alpha < b'. \end{cases}$$

Whence,

$$\left| \int_{\xi_{s-1}}^{\xi_s} \varphi dx \right| < 2\sigma \qquad \begin{cases} n > n_\sigma, \\ a' < \alpha < b', \end{cases}$$

and hence also (cf. (12)) we have for the value of α under consideration

$$(13) \qquad \left| \int_{-\pi}^{\xi_1-\omega} f \cdot \varphi dx \right| < 2Mp\sigma + B\rho; \qquad n > n_\sigma,$$

where M represents the upper limit of $|f(x)|$ in the intervals (8).

Similarly, all the $g + 1$ constituent integrals of S, except the last, may be thus treated, thereby leading to the equation

$$(14) \qquad S = P + \int_{x_g+\omega}^{\alpha-\epsilon} f \cdot \varphi dx,$$

where for all values of n greater than some value independent of α we have

$$|P| < 2gMp\sigma + gB\rho \leqq 2qMp\sigma + qB\rho.$$

Let us consider finally the integral appearing in (14). For this we first note that the interval of integration consists of a portion (or at most the whole) of the interval $(x_g + \omega, x_{g+1} - \omega)$ belonging to the set (8). Let us suppose that $\eta_r < \alpha - \epsilon \leqq \eta_{r+1}$ where η_r and η_{r+1} are the values of x corresponding to the extremities of the rth of the p divisions of length δ_{g+1} into which we have already divided the interval $(x_g + \omega, x_{g+1} - \omega)$. We may then write

$$\int_{x_g+\omega}^{\alpha-\epsilon} f \cdot \varphi dx = \int_{x_g+\omega}^{\eta_r} f \cdot \varphi dx + \int_{\eta_r}^{\alpha-\epsilon} f \cdot \varphi dx.$$

The last integral here appearing is less in absolute value (cf. relations (III) and (II)) than

$$(15) \qquad B \int_{\eta_r}^{\alpha-\epsilon} |f(x)| dx < B \int_{\eta_r}^{\eta_{r+1}} |f(x)| dx < B\rho$$

where ρ has the meaning already given.

Again, let there be l ($l \leqq p$) of the divisions δ_g in the interval $(x_g + \omega, \eta_r)$. Then, treating the first integral in the second member of (15) as we did the first integral in S, we obtain (cf. (13))

$$\left| \int_{x_g+\omega}^{\eta_r} f \cdot \varphi dx \right| < 2Ml\sigma + B\rho \leqq 2Mp\sigma + B\rho; \qquad n > n_\sigma,$$

where n_σ is independent of α.

In summary, then, we have the following result: Let $x_1, x_2, x_3, \cdots, x_s, \cdots, x_q$; $(x_s > x_{s-1})$; $(x_1 \mp -\pi,\ x_q \mp \pi)$ represent the q points within the interval $(-\pi, \pi)$ at which $f(x)$ becomes infinite, and let α be any value such that $\alpha - \epsilon$ (cf. (7)) lies within one of the intervals

$$(-\pi, x_1 - \omega), \qquad (x_1 + \omega, x_2 - \omega), \qquad \cdots, \qquad (x_q + \omega, \pi);$$

ω arbitrarily small and positive

and also such that $-\pi < a' \leqq \alpha \leqq b' < \pi$. Then, corresponding to an arbitrarily small positive quantity ρ and a second such quantity σ, we may determine a positive value n_σ *independent of* α and such that

$$\left| \int_{-\pi}^{\alpha-\epsilon} f(x)\varphi(n, x - \alpha)dx \right| < 2pM(q+1)\sigma + B(q+1)\rho; \qquad n > n_\sigma.$$

Since B, q, M and p as well as n_σ are each independent of α, it follows that for all the indicated values of α the first integral in the second member of (7) converges *uniformly* to zero when $n = \infty$.

It remains to show that the same is true when $\alpha - \epsilon$ pertains to one of the intervals of the following set:

$$(x_1 - \omega, x_1 + \omega), \qquad (x_2 - \omega, x_2 + \omega), \qquad \cdots, \qquad (x_q - \omega, x_q + \omega);$$

ω arb. small and positive.

The desired result follows by reasoning directly analogous to the preceding after rewriting (9) in which S and R are, however, defined as follows:

$$S = \left(\int_{-\pi}^{x_1-\omega} + \int_{x_1+\omega}^{x_2-\omega} + \cdots + \int_{x_{g-1}+\omega}^{x_g-\omega} \right) f(x)\varphi(n, x - \alpha)dx, \qquad (g \leqq q),$$

$$R = \left(\int_{x_1-\omega}^{x_1+\omega} + \int_{x_2-\omega}^{x_2+\omega} + \cdots + \int_{x_{g-1}-\omega}^{x_{g-1}+\omega} + \int_{x_g-\omega}^{\alpha-\epsilon} \right) f(x)\varphi(n, x - \alpha)dx.$$

Again, the same conclusion may be likewise reached in case either or both of the points $x = -\pi$, $x = \pi$ are points at which $f(x)$ becomes infinite. The forms in which S and R should then be taken readily suggest themselves and are therefore suppressed.

In like manner it appears that the second integral in the second member of (7) converges uniformly to zero when $n = \infty$ for all values of α such that

$$-\pi < a' \leqq \alpha \leqq b' < \pi.$$

These results having been established, we turn to a consideration of the last two integrals in the second member of (7). We shall suppose at first that α has any *special* value such that $-\pi < a' \leqq \alpha \leqq b' < \pi$.

Since by hypothesis $f(x)$ is of limited total fluctuation in the neighborhood of the point $x = \alpha$, the expressions $f(\alpha - 0)$, $f(\alpha + 0)$ certainly have a meaning.[9] We may therefore write the third integral in the second member of (7) in the form

$$(16) \qquad f(\alpha - 0)\int_{-\epsilon}^{0} \varphi(n, t)dt + \int_{-\epsilon}^{0} [f(\alpha + t) - f(\alpha - 0)]\varphi(n, t)dt.$$

When $n = \infty$ the first term here appearing approaches the limit $\frac{1}{2}f(\alpha - 0)$ as a result of relation (I). As to the second term, it follows from our hypotheses upon $f(x)$ in the neighborhood of the point $x = \alpha$ that the function $f(\alpha + t) - f(\alpha - 0)$ is of limited total fluctuation in the interval $-\epsilon < t < 0$, at least if ϵ be chosen sufficiently small. Whence, in this interval the same function will be either monotone or will consist of the difference of two monotone functions.[10] In the former case we may apply the second law of the mean for integrals and write

$$(17) \quad \int_{-\epsilon}^{0} [f(\alpha + t) - f(\alpha - 0)]\varphi(n, t)dt = [f(\alpha - \epsilon) + f(\alpha - 0)]\int_{-\epsilon}^{-\epsilon_1} \varphi(n, t)dt;$$
$$0 < \epsilon_1 < \epsilon.$$

At the same time our choice of ϵ may be made so small that the expression $|f(\alpha - \epsilon) - f(\alpha - 0)|$ will be less than any preassigned quantity σ. With ϵ thus chosen, we have now but to make use of relation (II) to see that the second term of (16) may be made less in absolute value than $A\sigma$, whatever the value of n. In case $f(\alpha + t) - f(\alpha - 0)$ consists of the difference of two monotone functions, the proof may evidently be carried out in a similar manner, showing that in this case the absolute value in question will be less than $2A\sigma$.

Therefore, the limit of the sum of the first and third terms in the second member of (7) as $n = \infty$ is $\frac{1}{2}f(\alpha - 0)$. Similarly, the limit approached by the sum of the second and last terms is $\frac{1}{2}f(\alpha + 0)$.

The first part of Theorem I is thus fully established. It remains to consider only that part which concerns *uniform* convergence, and since we have already shown that for all values of α such that $-\pi < a' \leqq \alpha \leqq b' < \pi$ the first and second terms in the second member of (7) converge uniformly to zero, it will

[9] Cf. HOBSON, *l. c.*, § 194.
[10] Cf. HOBSON, *l. c.*, § 195.

now suffice to show that under the hypotheses of the last part of the Theorem the last two terms of (7) when considered for the same values of α each converge uniformly to the limit $\frac{1}{2}f(\alpha)$.

Now, if $f(x)$ be continuous (as the present hypotheses demand) throughout the interval (a', b') ($x = a'$, $x = b'$ included) then $f(x)$ will be *uniformly* continuous throughout this interval.[11] Hence, corresponding to an arbitrary choice of

[11] Cf. HOBSON, *l. c.*, § 175.

the positive quantity σ, it is possible to determine a positive ϵ *independent of* α and such that

$$(18) \qquad |f(\alpha - \epsilon) - f(\alpha)| < \sigma; \qquad a' < \alpha < b'.$$

Introducing this choice of ϵ into (16), we may again write (17) for all the indicated values of α ($a' \leqq \alpha \leqq b'$) since, from the hypotheses of the second part of the Theorem, it follows as before that the function $f(\alpha + t) - f(\alpha)$ is either monotone or consists of the difference of two monotone functions of t throughout the interval $-\epsilon < t < 0$ whatever the value of α ($a' \leqq \alpha \leqq b'$)—at least if ϵ be taken so small that $a' - \epsilon > a_1$, where a_1 has the meaning given in the Theorem.

Thus, we reach the desired result respecting the third term in the second member of (7) and similarly, we reach the indicated result for its last term.

47. We turn to the proof of Theorem II. It is our purpose here to show that relations (I) and (III) of § 46 together with the following suffice for the proof:

(II)′ Having placed

$$(19) \qquad \Phi(n, t) = \frac{1}{n+1}\sum_{n=0}^{n} \varphi(n, t),$$

where $\varphi(n, t)$ is the trigonometric expression (5), we may write for a given value of the positive quantity ϵ and all subsequently chosen sufficiently large values of n

$$\int_{-\epsilon}^{\epsilon} |\Phi(n, t)|\, dt < C;$$

where C is a constant (independent of both n and ϵ).

In proving Theorem II we shall therefore substitute relation (II)′ for relation (II) of § 46, but we shall employ relations (I) and (III) as before.

Assuming first that α has any special value such that $-\pi < a' \leqq \alpha \leqq b' < \pi$, we have from (7)

$$(20) \qquad \begin{aligned} \frac{1}{n+1}[s_0(\alpha) + s_1(\alpha) + \cdots + s_n(\alpha)] &= \int_{-\pi}^{\alpha-\epsilon} f(x)\Phi(n, x-\alpha)dx \\ &+ \int_{\alpha+\epsilon}^{\pi} f(x)\Phi(n, x-\alpha)dx + \int_{\alpha-\epsilon}^{\alpha} f(x)\Phi(n, x-\alpha)dx + \int_{\alpha}^{\alpha+\epsilon} f(x)\Phi(n, x-\alpha)dx, \end{aligned}$$

where Φ is given by (19).

Now, the fact that φ satisfies (I) and (III) enables us to say at once that Φ also satisfies the same relations. In fact, if φ satisfies (I) the principle of consistency (§ 37) as applied to the HÖLDER method of summation shows that Φ also satisfies it, while (III) becomes satisfied by Φ since we may write

$$|\Phi(n, t)| \leqq \frac{1}{n}[|\varphi(n, t)| + |\varphi(n-1, t)| + \cdots + |\varphi(0, t)|] < \frac{nB}{n} = B.$$

The second member of (20) is the same as that of (7) except for the substitution of Φ for φ, and since Φ satisfies relations (I) and (III) it follows precisely as in the discussion in § 46 that the first two integrals on the right in (20), when considered for values of α such that $-\pi < a' \leqq \alpha \leqq b' < \pi$, converge uniformly to zero as $n = \infty$, provided merely that the integral (4) exists. The third term of (20) may be written in the form

$$f(\alpha - 0)\int_{-\epsilon}^{0} \Phi(n, t)dt + \int_{-\epsilon}^{0} [f(\alpha + t) - f(\alpha - 0)]\Phi(n, t)dt,$$

provided that $f(\alpha - 0)$ exists. When $n = \infty$ the first term here appearing approaches the limit $\frac{1}{2}f(\alpha - 0)$ since, as already pointed out, $\Phi(n, t)$ satisfies (I). As to the second term, we may choose ϵ so small that throughout the interval $-\epsilon < t < 0$ we shall have $|f(\alpha + t) - f(\alpha - 0)| < \sigma$ where σ is an arbitrarily small preassigned positive quantity. With ϵ thus chosen and n then taken sufficiently large the term in question becomes less in absolute value than

$$\sigma\int_{0}^{\epsilon} |\Phi(n, t)| dt < C\sigma, \tag{21}$$

where C is the constant defined in connection with relation (II)'. Thus, the sum of the first and third terms of (20) approaches the limit $\frac{1}{2}f(\alpha - 0)$ when $n = \infty$. Likewise, the sum of the second and last terms of (20) is seen to approach the limit $\frac{1}{2}f(\alpha + 0)$.

The first part of Theorem II thus becomes established, and in order to prove the second part it suffices to note (cf. the discussion of (16)) that if $f(x)$ is continuous throughout the interval (a', b') inclusive of the end points, then the quantity ϵ in (20) may be chosen independently of α $(a' < \alpha < b')$.

48. Having shown that Theorem I results from relations (I), (II) and (III) and that Theorem II results from (I), (II)' and (III), we shall now show that Theorem III results from (I), (II) and (III) together with the following:

$$\varphi(n, t \pm 2\pi) = \varphi(n, t). \tag{IV}$$ [12]

Let us take first the case in which $x = \pi$. The expression for $s_n(\pi)$ may be

[12] The proof of (IV) is immediate from (5).

obtained by placing $\alpha = \pi$ in (6). This expression, after placing $x - \pi = t$, becomes

$$(22)\qquad s_n(\pi) = \int_{-2\pi}^{0} f(\pi + t)\varphi(n, t)dt = \left(\int_{-2\pi}^{-\pi} + \int_{-\pi}^{0}\right) f(\pi + t)\varphi(n, t)dt.$$

Of the two integrals here appearing in the last member, the first, after making the substitution $t' = 2\pi + t$ and dropping accents, takes the following form as a result of (IV)

$$\int_{0}^{\pi} f(-\pi + t)\varphi(n, t)dt.$$

Whence, we may write

$$(23)\qquad \begin{aligned} s_n(\pi) = &\int_{-\pi}^{-\epsilon} f(\pi + t)\varphi(n, t)dt + \int_{\epsilon}^{\pi} f(-\pi + t)\varphi(n, t)dt \\ &+ \int_{-\epsilon}^{0} f(\pi + t)\varphi(n, t)dt + \int_{0}^{\epsilon} f(-\pi + t)\varphi(n, t)dt \end{aligned} \qquad (\epsilon > 0).$$

We may now show that as $n = \infty$ the limit approached by each of the first two integrals here appearing is 0. In order to do this it will suffice, since the integral (4) exists, to show that the property just indicated is true of each of the integrals

$$\int_{c}^{d} f(\pi + t)\varphi(n, t)dt, \qquad \int_{e}^{f} f(-\pi + t)\varphi(n, t)dt,$$

where it is understood that $f(\pi + t)$ remains finite throughout the (closed) interval (c, d); $-\pi \leqq c < d \leqq -\epsilon$, while $f(-\pi + t)$ remains finite throughout the (closed) interval (e, f); $\epsilon \leqq e < f \leqq \pi$.

Let us divide the interval (c, d) into p equal sub-intervals each of length δ by means of the points $t = c$, $t = t_1$, $t = t_2$, $\cdots$, $t = t_{p-1}$, $t = d$. Then, with the meaning for B appearing in (III), we may write

$$\int_{t_{s-1}}^{t_s} f(\alpha + t)\varphi(n, t)dt = \int_{t_{s-1}}^{t_s} f(\alpha + t)[\varphi(n, t) + B]dt - B\int_{t_{s-1}}^{t_s} f(\alpha + t)dt$$

and $[\varphi(n, t) + B]$ will be positive when $t_{s-1} < t < t_s$ (n having any fixed value). Hence, applying the first law of the mean for integrals, we obtain

$$\int_{t_{s-1}}^{t_s} f(\alpha + t)\varphi(n, t)dt = f_s\int_{t_{s-1}}^{t_s} \varphi(n, t)dt + Bf_s\int_{t_{s-1}}^{t_s} dt - Bf_s'\int_{t_{s-1}}^{t_s} dt,$$

where f_s and f_s' are certain quantities lying between the upper and lower limits of $f(\alpha + t)$ when $t_{s-1} < t < t_s$.

Since $t_s - t_{s-1} = \delta$, we thus have

$$\int_{t_{s-1}}^{t_s} f(\alpha + t)\varphi(n, t)dt = f_s\int_{t_{s-1}}^{t_s} \varphi(n, t)dt + \theta_s B\delta D_s; \qquad -1 < \theta_s < 1,$$

where D_s is the fluctuation of $f(\alpha + t)$ in (t_{s-1}, t_s).

Hence also

$$(24)\quad \int_c^d f(\alpha+t)\varphi(n,t)dt = \sum_{s=1}^{p} f_s \int_{t_{s-1}}^{t_s} \varphi(n,t)dt + \theta B\delta \sum_{s=1}^{p} D_s; \quad -1 < \theta < 1.$$

Now, by taking p sufficiently large the last term of (24) may be made arbitrarily small in absolute value, as follows from the existence of (4). The value of p having once been chosen, let us allow n to increase indefinitely. The last term of (24) continues arbitrarily small in absolute value, while its first term approaches the limit zero, as appears directly upon writing

$$\int_{t_{s-1}}^{t_s} = \int_0^{t_s} - \int_0^{t_{s-1}}$$

and applying (I).

Similarly, the second term in the second member of (23) is seen to have the property already indicated.

As to the third integral in the second member of (23), let us write

$$(25)\quad \begin{aligned} \int_{-\epsilon}^{0} f(\pi+t)\varphi(n,t)dt &= f(\pi-0)\int_{-\epsilon}^{0}\varphi(n,t)dt \\ &\quad + \int_{-\epsilon}^{0}[f(\alpha+t)-f(\pi-0)]\varphi(n,t)dt, \end{aligned}$$

noting that $f(\pi-0)$ necessarily exists since, according to the hypotheses in Theorem III, the function $f(x)$ is of limited total fluctuation in the neighborhood at the left of the point $x=\pi$. Upon comparing (25) with (16) and noting the statements in § 46 connected with the latter, we see at once that as $n = \infty$ the expression (25) approaches the limit $\frac{1}{2}f(\pi-0)$. Likewise, as $n = \infty$ the last term of (23) is seen to approach the limit $\frac{1}{2}f(-\pi+0)$.

In case $x=-\pi$ (instead of $x=\pi$) we have the following equations corresponding to (22) and (23):

$$\begin{aligned} s_n(-\pi) &= \int_0^{2\pi} f(-\pi+t)\varphi(n,t)dt = \left(\int_0^{\pi}+\int_{\pi}^{2\pi}\right)f(-\pi+t)\varphi(n,t)dt \\ &= \int_0^{\pi} f(-\pi+t)\varphi(n,t)dt + \int_{-\pi}^{0} f(\pi+t)\varphi(n,t)dt \end{aligned}$$

or

$$(26)\quad \begin{aligned} s_n(-\pi) &= \int_{-\pi}^{-\epsilon} f(\pi+t)\varphi(n,t)dt + \int_{\epsilon}^{\pi} f(-\pi+t)\varphi(n,t)dt \\ &\quad + \int_{-\epsilon}^{0} f(\pi+t)\varphi(n,t)dt + \int_0^{\epsilon} f(-\pi+t)\varphi(n,t)dt, \end{aligned}$$

and, upon considering the four integrals here appearing on the right as we considered those in (23), we find

$$\lim_{n=\infty} s_n(-\pi) = 0 + 0 + \tfrac{1}{2}f(\pi+0) + \tfrac{1}{2}f(-\pi+0).$$

Thus, the proof of Theorem III becomes complete.

49. Theorem IV likewise follows from (I), (II)′, (III) and (IV) upon noting that the expressions

$$\frac{1}{n+1}[s_0(\pm\pi)+s_1(\pm\pi)+\cdots+s_n(\pm\pi)]$$

may be obtained by replacing $\varphi(n, t)$ by $\Phi(n, t)$ in (23) and (26) and that, as a result of (IV), we have $\Phi(n, t\pm 2\pi)=\Phi(n, t)$.

II

The Representation of Arbitrary Functions by Means of Definite Integrals. The Formation of a General Theory for the Study of the Summability and Convergence of Fourier Series and Other Allied Developments

50. The manner in which the summability and convergence of Fourier series has been shown in §§ 47–49 to depend upon the properties of the integrals

$$\int \varphi(n, t)dt, \qquad \int \Phi(n, t)dt,$$

where $\varphi(n, t)$ and $\Phi(n, t)$ are defined by (5) and (19) readily suggests the general problem of determining a set of sufficient conditions for *any* function $\varphi(n, t)$ of the two real variables n, t, or more generally, for any function $\varphi(n, \alpha, t)$ of the *three* real variables n, α, t in order that the integral (cf. (6))

$$(27)\qquad \int_{a-\alpha}^{b-\alpha} f(\alpha+t)\varphi(n, \alpha, t)dt \qquad \text{or} \qquad \int_a^b f(x)\varphi(n, \alpha, x-\alpha)dx$$

shall converge when $n=\infty$ to the values $\frac{1}{2}[f(\alpha-0)+f(\alpha+0)]$ or $\frac{1}{2}[f(b-0)+f(a+0)]$ according as $a<\alpha<b$ or $\alpha=$ either a or b. Naturally, the range of possible existence for such functions φ will depend upon the conditions imposed upon the given function $f(x)$ when considered throughout the interval (a, b), and in determining the form of such conditions we shall hereafter be guided by the limitations upon $f(x)$ occurring in the Theorems of § 46. The general theorems about to be obtained will serve as a foundation for the discussion in §§ 64–70 relative to the summability and convergence of the well-known developments in terms of Bessel functions, Legendre functions, etc.

51. Theorem I. *Let $\varphi(n, \alpha, t)$ be a function of the real variables n, α, t which, when considered for values of α lying within any sub-interval (a', b') of (a, b) $(a<a'<b'<b)$ satisfies the following three relations in which n is restricted to positive integral values and in which ϵ represents a positive quantity which may be taken arbitrarily small:*

$$(\mathrm{I})\qquad \lim_{n=\infty}\int_0^t \varphi(n, \alpha, t)dt=\begin{cases}-\frac{1}{2} & \textit{when}\quad a-\alpha\leqq t\leqq-\epsilon;\\ \ \ \frac{1}{2} & \textit{when}\quad \epsilon\leqq t\leqq b-\alpha.\end{cases}$$

Moreover, let these limits be approached uniformly for all of the same values of α and t.[13]

$$\text{(II)} \qquad \left| \int_0^t \varphi(n, \alpha, t)dt \right| < A; \qquad -\epsilon \leqq t \leqq \epsilon,$$

where A represents a constant independent of n, α and t.

$$\text{(III)} \qquad |\varphi(n, \alpha, t)| < B; \qquad a - \alpha \leqq t \leqq -\epsilon \quad or \quad \epsilon \leqq t \leqq b - \alpha,$$

where B represents a constant independent of n, α and t.

Also, let $f(x)$ be any function satisfying the following two conditions:

(A) When considered throughout the interval (a, b), $f(x)$ remains finite with the possible exception of a finite number of points and is such that the integral

$$\int_a^b |f(x)|dx$$

exists.

(B) When considered in an arbitrarily small neighborhood about the (special) point $x = \alpha$ $(a' < \alpha < b')$ $f(x)$ has limited total fluctuation.

Then we shall have for the (special) value of α mentioned in (B)

$$\text{(28)} \qquad \lim_{n=\infty} \int_a^b f(x)\varphi(n, \alpha, x - \alpha)dx = \tfrac{1}{2}[f(\alpha - 0) + f(\alpha + 0)].$$

Moreover, if (instead of condition (B)) $f(x)$ is continuous throughout the interval (a', b'), the points $x = a'$, $x = b'$ included, and has limited total fluctuation throughout an interval (a_1, b_1) such that $a < a_1 < a' < b' < b_1 < b$, then we shall have uniformly for all values of α in (a', b')

$$\text{(29)} \qquad \lim_{n=\infty} \int_a^b f(x)\varphi(n, \alpha, x - \alpha)dx = f(\alpha).$$

[13] Thus, to an arbitrarily small positive quantity σ it shall be possible to determine a value n_σ independent of both α and t such that

$$\left| \int_0^t \varphi(n, \alpha, t)dt + \tfrac{1}{2} \right| < \sigma; \qquad n > n_\sigma$$

provided α and t are assigned values consistently with the relations

$$a' < \alpha < b'; \qquad a - \alpha \leqq t \leqq -\epsilon.$$

Likewise,

$$\left| \int_0^t \varphi(n, \alpha, t)dt - \tfrac{1}{2} \right| < \sigma; \qquad n > n_\sigma$$

provided α and t are assigned values consistently with the relations

$$a' < \alpha < b'; \qquad \epsilon \leqq t \leqq b - \alpha.$$

It may be added that in case one confines the attention to the convergence of the integral (27) for *special* values of α (thus not considering questions of uniform convergence) it suffices that relation (I) shall be satisfied for each special value of α $(a' < \alpha < b')$. Similarly, the constants A and B of (II) and (III) may then depend upon α.

Proof.—The proof of this theorem is readily supplied upon referring to the methods employed in § 46 for the study of the integral (6). We shall therefore merely indicate the essential steps.

Representing the integral (27) by $s_n(\alpha)$, we first write (cf. (7))

$$s_n(\alpha) = \int_a^{\alpha-\epsilon} f(x)\varphi(n, \alpha, x - \alpha)dx + \int_{\alpha+\epsilon}^b f(x)\varphi(n, \alpha, x - \alpha)dx$$
$$+ \int_{\alpha-\epsilon}^{\alpha} f(x)\varphi(n, \alpha, x - \alpha)dx + \int_{\alpha}^{\alpha+\epsilon} f(x)\varphi(n, \alpha, x - \alpha)dx.$$

Of the four integrals here appearing, the first two approach the limit zero as $n = \infty$ and the convergence is uniform for all values of α such that $a' < \alpha < b'$, as results from (I), (III) and (*A*). Moreover, the third and fourth integrals (considered for any special value of α such that $a' < \alpha < b'$) approach respectively the limits $\frac{1}{2}f(\alpha - 0)$, $\frac{1}{2}f(\alpha + 0)$, as results from (I), (II) and (*B*) (cf. (17), (18)).

Likewise, upon comparison with the corresponding studies in § 46, it appears that equation (29) will hold true *uniformly* under the conditions stated in the theorem.

52. THEOREM II. *Let $\varphi(n, \alpha, t)$ be a function of the real variables n, α, t which, when considered for values of α such that $a < a' < \alpha < b' < b$, satisfies relations* (I) *and* (III) *of* § 51 *and is such that if we place*

$$(30) \qquad \Phi(n, \alpha, t) = \frac{1}{n+1}[\varphi(n, \alpha, t) + \varphi(n - 1, \alpha, t) + \cdots + \varphi(0, \alpha, t)]$$

the following relation is satisfied: corresponding to a given $\epsilon > 0$ we shall have for all subsequently chosen sufficiently large values of n

$$\text{(II)}' \qquad \int_{-\epsilon}^{\epsilon} |\Phi(n, \alpha, t)|dt < C$$

where C represents a constant independent of n, α and ϵ.

Also, let $f(x)$ be any function which satisfies condition (*A*) *of* § 51 *together with the following:*

(*B*)′ *When considered in the neighborhood of the* (*special*) *point $x = \alpha$ ($a' < \alpha < b'$), the limits $f(\alpha - 0)$, $f(\alpha + 0)$ exist.*

Then we shall have for the (*special*) *value of α mentioned in* (*B*)′

$$(31) \qquad \lim_{n=\infty} \int_a^b f(x)\Phi(n, \alpha, x - \alpha)dx = \tfrac{1}{2}[f(\alpha - 0) + f(\alpha + 0)].$$

Moreover, if (*instead of condition* (*B*)′) *$f(x)$ is continuous throughout the interval (a', b'), the points $x = a'$, $x = b'$ included, we shall have uniformly for all of the same values of α*

$$\lim_{n=\infty} \int_a^b f(x)\Phi(n, \alpha, x - \alpha)dx = f(\alpha).$$

The proof of this theorem, like that just indicated for Theorem I, is at once supplied upon following the steps indicated in § 46 with reference to the special integral (6) there occurring. We therefore omit it.

53. As a generalization of the Theorem III of § 46 we have the following

THEOREM III. *Let $\varphi(n, \alpha, t)$ be a function of the real variables n, α, t which, when considered for the special values $\alpha = a$, $\alpha = b$ $(b > a)$ satisfies the following four relations in which n is restricted to positive integral values and in which ϵ represents a positive quantity which may be taken arbitrarily small:*

$$(\mathrm{I})_{a,b}\quad \begin{cases} \lim\limits_{n=\infty} \displaystyle\int_0^t \varphi(n, a, t)dt = -\tfrac{1}{2} & \textit{when} \quad a - b + \epsilon \leqq t \leqq -\epsilon, \\ \lim\limits_{n=\infty} \displaystyle\int_0^t \varphi(n, b, t)dt = \tfrac{1}{2} & \textit{when} \quad \epsilon \leqq t \leqq b - a - \epsilon. \end{cases}$$

$(\mathrm{II})_{a,b}$ *Relation* (II) *of* § 55 *is satisfied when $\alpha = a$ and t lies in the interval* $0 \leqq t \leqq \epsilon$; *also when $\alpha = b$ and t lies in the interval $-\epsilon \leqq t \leqq 0$.*

$$(\mathrm{III})_{a,b}\quad \begin{cases} |\varphi(n, a, t)| < B & \textit{when} \quad a - b + \epsilon \leqq t \leqq -\epsilon, \\ |\varphi(n, b, t)| < B & \textit{when} \quad \epsilon \leqq t \leqq b - a - \epsilon, \end{cases}$$

where B is a constant independent of both n and t.

$$(\mathrm{IV})\qquad \varphi(n, a, t + b - a) = \varphi(n, b, t + b - a) = \varphi(n, a, t) = \varphi(n, b, t).$$

Also, let $f(x)$ be any function which satisfies condition (A) of § 51 and is such that in arbitrarily small neighborhoods to the right of the point $x = a$ and to the left of the point $x = b$ it has limited total fluctuation.

Then we shall have

$$\lim_{n=\infty} \int_a^b f(x)\varphi(n, a, x - a)dx = \lim_{n=\infty} \int_a^b f(x)\varphi(n, b, x - b)dx$$
$$= \tfrac{1}{2}[f(b - 0) + f(a + 0)].$$

Proof.—Here again the proof may be easily supplied upon reference to the analysis occurring in § 48. Thus, for the case in which $\alpha = b$ we may write

$$s_n(b) = \int_{a-b}^0 f(b + t)\varphi(n, b, t)dt = \left(\int_{a-b}^{a-b+\epsilon} + \int_{a-b+\epsilon}^{-\epsilon} + \int_{-\epsilon}^0\right) f(b + t)\varphi(n, b, t)dt,$$

which, upon making the transformation $t' = b - a + t$ in the first integral of the last member and making use of (IV) becomes

$$s_n(b) = \int_{a-b+\epsilon}^{-\epsilon} f(b + t)\varphi(n, b, t)dt + \int_{-\epsilon}^0 f(b + t)\varphi(n, b, t)dt$$
$$+ \int_0^\epsilon f(a + t)\varphi(n, a, t)dt.$$

Of the three integrals here appearing, the first approaches the limit zero when $n = \infty$, as results from $(\mathrm{I})_{a,b}$, $(\mathrm{III})_{a,b}$ and (A) while the second and third approach respectively the limits $\frac{1}{2}f(b-0)$ and $\frac{1}{2}f(a+0)$, as results from $(\mathrm{I})_{a,b}$, $(\mathrm{II})_{a,b}$ and the assumption regarding the behavior of $f(x)$ in neighborhoods arbitrarily near to the points $x = a$ and $x = b$.

Similarly, in case $\alpha = a$ we may write

$$s_n(a) = \int_0^{b-a} f(a+t)\varphi(n, a, t)dt = \left(\int_0^{\epsilon} + \int_{\epsilon}^{b-a-\epsilon} + \int_{b-a-\epsilon}^{b-a}\right) f(a+t)\varphi(n, a, t)dt$$

$$= \int_{\epsilon}^{b-a-\epsilon} f(a+t)\varphi(n, a, t)dt + \int_{-\epsilon}^{0} f(b+t)\varphi(n, b, t)dt$$

$$+ \int_0^{\epsilon} f(a+t)\varphi(n, a, t)dt,$$

from which we deduce the indicated result as before.

54. Again, we have (cf. the remarks in § 49 on Theorem IV) the following

THEOREM IV. *Let $\varphi(n, \alpha, t)$ be a function of the real variables n, α, t which, when considered for the special values $\alpha = a$ and $\alpha = b$ satisfies relations* $(\mathrm{I})_{a,b}$, $(\mathrm{III})_{a,b}$ *and* (IV) *of* § 54 *and also the following:*

The integrals

$$(\mathrm{II})'_{a,b} \qquad \int_0^{\epsilon} |\Phi(n, -1, t)|dt, \qquad \int_{-\epsilon}^{0} |\Phi(n, 1, t)|dt \qquad (\epsilon > 0),$$

when considered for all values of n sufficiently large remain less that a constant (independent of ϵ).

Also, let $f(x)$ be any function which satisfies condition (A) *of* § 51 *and is such that the limits $f(a+0)$, $f(b-0)$ exist.*

Then we shall have

$$\lim_{n=\infty} \int_a^b f(x)\Phi(n, a, x-a)dx = \lim_{n=\infty} \int_a^b f(x)\Phi(n, b, x-b)dx$$

$$= \tfrac{1}{2}[f(b-0) + f(a+0)].$$

55. Besides the relations given in Theorems III and IV concerning the functions $\varphi(n, a, t)$ and $\varphi(n, b, t)$ (which relations are satisfied in particular by the function (5) pertaining to Fourier series, with $\alpha = -\pi$ or $\alpha = \pi$) it is important to note certain others which we shall find fulfilled by some of the functions $\varphi(n, \alpha, t)$ met with in the succeeding pages but which are not fulfilled by (5). These relations together with their effects upon the limiting values of the integrals

$$\int_a^b f(x)\varphi(n, \alpha, x-\alpha)dx, \qquad \int_a^b f(x)\Phi(n, \alpha, x-\alpha)dx$$

we now summarize in the following four theorems:

THEOREM V. *Let $\varphi(n, \alpha, t)$ be a function of the real variables n, α, t which, when considered for the special value $\alpha = a$ satisfies the following three relations in which n is restricted to positive integral values and in which ϵ represents a positive quantity which may be taken arbitrarily small:*

$$(\mathrm{I})_a \qquad \lim_{n=\infty} \int_0^t \varphi(n, a, t)dt = G_1; \qquad \epsilon \leqq t \leqq b - a \qquad (b > a),$$

G_1 being a constant (independent of t).
$(\mathrm{II})_a$ *Relation* (II) *of* § 51 *is satisfied when $\alpha = a$ and $0 \leqq t \leqq \epsilon$.*

$$(\mathrm{III})_a \qquad |\varphi(n, a, t)| < B; \qquad \epsilon \leqq t \leqq b - a,$$

B being a constant independent of n and t.

Also, let $f(x)$ be any function which satisfies condition (A) *of* § 51 *and is such that it has limited total fluctuation in an arbitrarily small neighborhood at the right of the point $x = a$.*

Then we shall have

$$\lim_{n=\infty} \int_a^b f(x)\varphi(n, a, x - a)dx = G_1 f(a + 0).$$

THEOREM VI. *Let $\varphi(n, \alpha, t)$ be a function of the real variables n, α, t which, when considered for the special value $\alpha = b$ satisfies the following three relations in which n is restricted to positive integral values and in which ϵ represents a positive quantity which may be taken arbitrarily small:*

$$(\mathrm{I})_b \qquad \lim_{n=\infty} \int_0^t \varphi(n, b, t)dt = -G_2; \qquad a - b \leqq t \leqq -\epsilon \qquad (b > a),$$

G_2 being a constant (independent of t).
$(\mathrm{II})_b$ *Relation* (II) *of* § 51 *is satisfied when $\alpha = b$ and $-\epsilon \leqq t \leqq 0$.*

$$(\mathrm{III})_b \qquad |\varphi(n, b, t)| < B; \qquad a - b \leqq t \leqq -\epsilon,$$

B being a constant independent of both n and t.

Also, let $f(x)$ be any function which satisfies condition (A) *of* § 51 *and is such that it has limited total fluctuation in an arbitrarily small neighborhood at the left of the point $x = b$.*

Then we shall have

$$\lim_{n=\infty} \int_a^b f(x)\varphi(n, b, x - b)dx = G_2 f(b - 0).$$

THEOREM VII. *Let $\varphi(n, \alpha, t)$ be a function satisfying relations* $(\mathrm{I})_a$ *and* $(\mathrm{III})_a$ *of Theorem V but, instead of* $(\mathrm{II})_a$, *the following:*

$(\mathrm{II})_a'$ *Relation* (II)$'$ *of* § 54 *is satisfied when $\alpha = a$, it being understood that the integration there appearing is then taken from 0 to ϵ instead of from $-\epsilon$ to ϵ.*

Also, let $f(x)$ be any function which satisfies condition (A) *of* § 51 *and is such that the limit* $f(a+0)$ *exists.*

Then we shall have

$$\lim_{n=\infty}\int_a^b f(x)\Phi(n,\,a,\,x-a)dx = G_1 f(a+0),$$

where Φ *is defined by* (30).

THEOREM VIII. *Let* $\varphi(n,\alpha,t)$ *be a function satisfying relations* $(\mathrm{I})_b$ *and* $(\mathrm{III})_b$ *of Theorem VI but, instead of* $(\mathrm{II})_b$, *the following:*

$(\mathrm{II})_b'$ *Relation* (II)$'$ *of* § 52 *is satisfied when* $\alpha = b$, *it being understood that the integration there appearing is then taken from* $-\epsilon$ *to* 0 *instead of from* $-\epsilon$ *to* ϵ.

Also, let $f(x)$ be any function which satisfies condition (A) *of* § 51 *and is such that the limit* $f(b-0)$ *exists.*

Then we shall have

$$\lim_{n=\infty}\int_a^b f(x)\Phi(n,\,b,\,x-b)dx = G_2 f(b-0),$$

where Φ *is defined by* (30).

The first of the Theorems V, VI results directly upon writing

$$s_n(a) = \int_0^{b-a} f(a+t)\varphi(n,\,a,\,t)dt = \left(\int_0^{\epsilon} + \int_{\epsilon}^{b-a}\right) f(a+t)\varphi(n,\,a,\,t)dt; \qquad \epsilon > 0$$

and then applying to each of the last two integrals the methods already used in § 48 for the study of similar integrals.

Theorem VI likewise results upon writing

$$(32)\qquad s_n(b) = \int_{a-b}^{0} f(b+t)\varphi(n,\,b,\,t)dt = \left(\int_{-\epsilon}^{0} + \int_{a-b}^{-\epsilon}\right) f(b+t)\varphi(n,\,b,\,t)dt.$$

The proofs of Theorems VII and VIII being likewise readily supplied, are suppressed.

56. We proceed to make certain observations which will prove useful in applying the general theorems of §§ 51–55 to *special* integrals (27).

(1) If in applying Theorem I of § 51 it is found that for some special value of t different from zero, $t = t_1 \neq 0$ say, the function $\varphi(n,\alpha,t)$ becomes infinite or otherwise is of such a character that uncertainty arises concerning any one of the relations (I), (II), (III) when $t = t_1$, then the theorem will still hold good provided that it can be shown that the integral

$$I_\xi = \int_{t_1-\xi}^{t_1+\xi} |f(\alpha+t)\varphi(n,\,\alpha,\,t)|dt,$$

where ξ is arbitrarily small and > 0, approaches $(n=\infty)$ uniformly the limit zero for $a < a' \leqq \alpha \leqq b' < b$, or else is such that for the same values of α and

for all (positive integral) values of n the same integral approaches uniformly the limit zero as $\xi = 0$.

An examination of the method used in proving Theorem I shows at once the correctness of this remark. More generally, in case of uncertainty of any kind in the behavior of $f(\alpha + t)\varphi(n, \alpha, t)$ for the value $t = t_1 \mp 0$ $(a - \alpha < t_1 < b - \alpha)$, it suffices for the existence of (28) and (29) that relations (I), (II), (III), (A) and (B) (or in (29) the substitute for (B) there mentioned) shall be satisfied throughout the two intervals $(a - \alpha \leqq t \leqq t_1 - \xi)$, $(t_1 + \xi \leqq t \leqq b - \alpha)$ (ξ arbitrarily small and positive) instead of throughout the whole interval $(a - \alpha, b - \alpha)$, provided merely that the expression I_ξ above defined has either of the properties just mentioned.

If the exceptional point is $t_1 = a - \alpha$ then, instead of the two intervals, we have to consider the single one $(a - \alpha + \xi \leqq t \leqq b - \alpha)$, while instead of I_ξ as defined above, we shall have to consider the integral

$$I_\xi^{(a)} = \int_{a-\alpha}^{a-\alpha+\xi} |f(\alpha + t)\varphi(n, \alpha, t)|\,dt.$$

A corresponding statement may at once be supplied for the case in which the exceptional point is $t_1 = b - \alpha$.

In the case of two or more of the exceptional points t_1 $(a - \alpha \leqq t_1 \leqq b - \alpha)$ the corresponding statements are readily supplied.

(2) The conditions demanded in Theorem II may be stated without reference to the function $\varphi(n, \alpha, t)$. Thus, it suffices (aside from the conditions upon $f(x)$) that the function $\Phi(n, \alpha, t)$ shall satisfy relations (I) and (III) of Theorem I together with (II)′ of Theorem II.

This follows from the fact that the conditions placed upon $\varphi(n, \alpha, t)$ in Theorem II are there inserted merely that $\Phi(n, \alpha, t)$ may have the properties just indicated, the latter being those upon which the proof in reality depends.

Similarly, in using Theorems VII and VIII the conditions stated relative to $\varphi(n, a, t)$, $\varphi(n, b, t)$ may be replaced by the same conditions referred to $\Phi(n, a, t)$, $\Phi(n, b, t)$.

(3) Assuming that relations (II), (III), (A) and (B) of Theorem I are satisfied, let us suppose that instead of relation (I) we have the following:[14]

$$\text{(I)}' \qquad \lim_{n=\infty} \int_0^t \varphi(n, \alpha, t)dt = \begin{cases} -\tfrac{1}{2} + \chi(\alpha, t) & \textit{when} \quad a - \alpha \leqq t \leqq -\epsilon, \\ \tfrac{1}{2} + \chi(\alpha, t) & \textit{when} \quad \epsilon \leqq t \leqq b - \alpha, \end{cases}$$

where $\chi(\alpha, t)$ is any function of α and t such that

(a) Having given an arbitrarily small positive quantity σ, one may determine a positive quantity ξ dependent only upon σ such that

[14] As in (I) of § 51, it is here to be understood that the convergence ($n = \infty$) is uniform for the indicated values of α and t.

$$|\chi(\alpha, t)| < \sigma \quad when \quad \begin{cases} a' \leqq \alpha \leqq b', \\ -\xi \leqq t \leqq \xi. \end{cases}$$

(b) The partial derivative $\partial\chi/\partial t$ exists whenever $a' \leqq \alpha \leqq b'$, $a - \alpha \leqq t \leqq b - \alpha$ and for the same values of α and t is such that

$$\left|\frac{\partial\chi}{\partial t}\right| < D = \textit{constant independent of } \alpha \textit{ and } t.$$

Under these conditions it is easily seen that the function $\varphi(n, \alpha, t) - \partial\chi/\partial t$ comes to satisfy relations (I), (II) and (III) of the theorem of § 51 from which it follows that for a fixed value of α such that $a' \gtreqless \alpha \gtreqless b'$ we may write

$$-\int_a^b f(x)\left[\frac{\partial\chi}{\partial t}\right]_{t=x-\alpha} dx + \lim_{n\to\infty}\int_a^b f(x)\varphi(n, \alpha, x-\alpha)dx = \frac{f(\alpha-0)+f(\alpha+0)}{2}.$$

Moreover, if (instead of condition (B)) $f(x)$ is continuous throughout the interval $a' \leqq x \leqq b'$, the end points $x = a'$, $x = b'$ included, and has limited total fluctuation throughout an interval (a_1, b_1) such that $a < a_1 < a' < b' < b_1 < b$, then for all values of α in (a', b') the equation will hold true uniformly, it being understood that the right member is then replaced by $f(\alpha)$.

Analogous remarks relative to Theorems (III), (V), (VI) are readily supplied.

III

The Calculus of Residues as Applied to the Series Developments for an Arbitrary Function.[15] The General Problem of Sturm

57. A comparison of the developments occurring in mathematical physics for a function $f(x)$ of one real variable x shows that they are ordinarily of the form

$$
(33) \quad \sum_{n=1}^{\infty}\left\{ H_1(\lambda_n, x)\frac{\int_a^b f(x)F(x)H_1(\lambda_n, x)dx}{\int_a^b F(x)H_1^2(\lambda_n, x)dx} + H_2(\lambda_n, x)\frac{\int_a^b f(x)F(x)H_2(\lambda_n, x)dx}{\int_a^b F(x)H_2^2(\lambda_n, x)dx} + \cdots + H_m(\lambda_n, x)\frac{\int_a^b f(x)F(x)H_m(\lambda_n, x)dx}{\int_a^b F(x)H_m^2(\lambda_n, x)dx}\right\},
$$

where $H_1(\lambda_n, x)$, $H_2(\lambda_n, x)$, $\cdots$, $H_m(\lambda_n, x)$ are m functions of x and of a certain parameter λ which takes different values from term to term in (33) according to some given law, and where $F(x)$ is a function of x only which is finite throughout the interval (a, b).

Thus in the case of a Fourier series we have $m = 2$, $H_1(\lambda_n, x) = \sin nx$, $H_2(\lambda_n, x) = \cos nx$; and $a = -\pi$, $b = \pi$, $F(x) = 1$. Again, in dealing with the usual expansion of $f(x)$ in terms of Bessel's function of order zero, we have $m = 1$, $H_1(\lambda_n, x) = J_0(\lambda_n, x)$, $a = 0$, $b = 1$, $F(x) = x$, λ_n being one of the roots of the transcendental equation $J_0(x) = 0$.

It is to the important developments (33) that we shall hereafter devote our attention.

The first n terms of (33) when considered for any particular value of x such as $x = \alpha$ may evidently be put into the form

$$
\int_a^b f(x)\varphi(n, \alpha, x - \alpha)dx,
$$

where

[15] The calculus of residues was first applied by Cauchy to the study of infinite series, in particular to Fourier series (cf. Picard, "Traité d'Analyse," Vol. II, Chap. VI, § 9 *et seq*). Its application to the general study of developments in terms of normal functions appears to have

(34) $$\varphi(n, \alpha, x - \alpha) = \sum_{r=1}^{n} \sum_{s=1}^{m} H_s(\lambda_r, \alpha) \frac{F(x)H_s(\lambda_r, x)}{\int_a^b F(x)H_s^2(\lambda_r, x)dx}.$$

Upon referring to the theorems of §§ 51–55 it thus appears that in order to show the summability or convergence of series (33) to the value

$$\frac{f(\alpha - 0) + f(\alpha + 0)}{2} \quad \text{or} \quad \frac{f(a + 0) + f(b - 0)}{2} \quad \text{or} \quad G_1 f(a + 0) \quad \text{or} \quad G_2 f(b - 0)$$

according to the cases there considered it suffices to show that the conditions specified for $\varphi(n, \alpha, t)$ in the same theorems are present when

(35) $$\varphi(n, \alpha, t) = \sum_{r=1}^{n} \sum_{s=1}^{m} H_s(\lambda_r, \alpha) \frac{F(\alpha + t)H_s(\lambda_r, \alpha + t)}{\int_a^b F(t)H_s^2(\lambda_r, t)dt}.$$

Thus the integral

(36) $$\int_0^t \varphi(n, \alpha, t)dt,$$

which plays an important part in these theorems, becomes in the present case

(37) $$\int_0^t \varphi(n, \alpha, t)dt = \sum_{r=1}^{n} \sum_{s=1}^{m} H_s(\lambda_r, \alpha) \frac{\int_0^t F(\alpha + t)H_s(\lambda_r, \alpha + t)dt}{\int_a^b F(t)H_s^2(\lambda_r, t)dt}.$$

58. Now, the values $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n, \cdots$ of the parameter λ are usually given as the roots (or part of the roots) of some transcendental equation $u(z) = 0$ where $u(z)$ is a function of the complex variable z which is analytic throughout all finite portions of the z plane. Thus in the case of Fourier's series we have $u(z) = \sin \pi z$ and in the above mentioned case of the expansion in Bessel's function of order zero we have $u(z) = J_0(z)$. Moreover, these roots, when considered as zeros of the function $u(z)$ are ordinarily zeros of the *first* order and we shall suppose this to be the case in what follows.

Then the function $w(z) = 1/u(z)$ will be analytic throughout the finite z plane with the exception of the points $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n, \cdots$, where it will have poles of the first order and, considering $\theta(z)$ to be any other function of z which is analytic throughout the finite z-plane, we shall have, provided p is a positive integer,

(38) $$\begin{aligned} \theta(z)w^p(z)(z - \lambda_n)^p = \theta(\lambda_n)A^p + [\theta(z)w^p(z)(z - \lambda_n)^p]^1_{\lambda_n}(z - \lambda_n) + \cdots \\ + \frac{[\theta(z)w^p(z)(z - \lambda_n)^p]^{p-1}_{\lambda_n}}{(p-1)!}(z - \lambda_n)^{p-1} + w_1(z)(z - \lambda_n)^p, \end{aligned}$$

where A is the limit of $w(z)(z-\lambda_n)$ as $z = \lambda_n$ and $w_1(z)$ is a function
is analytic in the neighborhood of the point $z = \lambda_n$ and where

$$[\theta(z)w^p(z)(z-\lambda_n)^p]^s_{\lambda_n}$$

indicates the value of the sth derivative of $\theta(z)w^p(z)(z-\lambda_n)^p$ at the po

From (38) we have

$$\theta(z)w^p(z) = \frac{\theta(\lambda_n)A^p}{(z-\lambda_n)^p} + \frac{[\theta(z)w^p(z)(z-\lambda_n)^p]^1_{\lambda_n}}{(z-\lambda_n)^{p-1}} + \cdots \tag{39}$$

$$+ \frac{[\theta(z)w^p(z)(z-\lambda_n)^p]^{p}_{\lambda}}{(z-\lambda_n)(p-1)!}$$

and integrating in the positive direction about any closed contou
encloses the point $z = \lambda_n$ but no other pole of $w(z)$ we have by Cauch
theorem

$$\frac{1}{2\pi i}\int_C \theta(z)w^p(z)dz = \frac{[\theta(z)w^p(z)(z-\lambda_n)^p]^{p-1}_{\lambda_n}}{(p-1)!}. \tag{40}$$

If, therefore, we integrate about a closed contour C_n which encloses
of the points $\lambda_1, \lambda_2, \lambda_3, \cdots$ but no other poles of $w(z)$ we shall have

$$\frac{1}{2\pi i}\int_{C_n} \theta(z)w^p(z)dz = \sum_{n=1}^{n}\frac{[\theta(z)w^p(z)(z-\lambda_n)^p]^{p-1}_{\lambda_n}}{(p-1)!} \tag{41}$$

and hence also

$$\lim_{n=\infty}\frac{1}{2\pi i}\int_{C_n} \theta(z)w^p(z)dz = \sum_{n=1}^{\infty}\frac{[\theta(z)w^p(z)(z-\lambda_n)^p]^{p-1}_{\lambda_n}}{(p-1)!} \tag{42}$$

whenever either side of the same relation has a meaning.

In particular, when $p = 1$ and $p = 2$ we have respectively

$$\frac{1}{2\pi i}\int_{C_n} \theta(z)w(z)dz = \sum_{n=1}^{n}[\theta(z)w(z)(z-\lambda_n)]_{\lambda_n}, \tag{43}$$

$$\frac{1}{2\pi i}\int_{C_n} \theta(z)w^2(z)dz = \sum_{n=1}^{n}[\theta(z)w^2(z)(z-\lambda_n)^2]^1_{\lambda_n}, \tag{44}$$

or, since

$$w(z)(z-\lambda_n) = \frac{1}{\dfrac{u(z)}{z-\lambda_n}}$$

$$= \frac{1}{u'(\lambda_n) + \left(\dfrac{u(z)}{z-\lambda_n}\right)'_{\lambda_n}(z-\lambda_n) + \dfrac{1}{2!}\left(\dfrac{u(z)}{z-\lambda_n}\right)''_{\lambda_n}(z-\lambda_n}$$

$$[\theta(z)w^2(z)(z-\lambda_n)^2]^1_{\lambda_n} = \theta'(\lambda_n)[w^2(\lambda_n)(z-\lambda_n)^2] + \theta(\lambda_n)[w^2(z)(z-\lambda_n)^2]^1_{\lambda_n}$$

$$= \frac{\theta'(\lambda_n)}{u'(\lambda_n)^2} - \theta($$

relations (43) and (44) may be written in the form

$$\frac{1}{2\pi i}\int_{C_n}\frac{\theta(z)}{u(z)}\,dz=\sum_{n=1}^{n}\frac{\theta(\lambda_n)}{u'(\lambda_n)},\tag{45}$$ [16]

$$\frac{1}{2\pi i}\int_{C_n}\frac{\theta(z)}{u^2(z)}\,dz=\sum_{n=1}^{n}\left\{\frac{\theta'(\lambda_n)}{u'(\lambda_n)}-\frac{\theta(\lambda_n)u''(\lambda_n)}{u'(\lambda_n)^3}\right\}.\tag{46}$$

It is desirable to note also that if in (46) we substitute $\theta(z)\psi(z)$ for $\theta(z)$ we obtain

$$\frac{1}{2\pi i}\int_{C_n}\frac{\theta(z)\psi(z)}{u^2(z)}\,dz=\sum_{n=1}^{n}\left\{\frac{\theta'(\lambda_n)\psi(\lambda_n)}{u'(\lambda_n)^2}+\theta(\lambda_n)\frac{\psi'(\lambda_n)u'(\lambda_n)-\psi(\lambda_n)u''(\lambda_n)}{u'(\lambda_n)^3}\right\},$$

so that if $\psi'(\lambda_n)u'(\lambda_n)-\psi(\lambda_n)u''(\lambda_n)=0$ we shall have

$$\frac{1}{2\pi i}\int_{C_n}\frac{\theta(z)}{u^2(z)}\,dz=\sum_{n=1}^{n}\frac{\theta'(\lambda_n)\psi(\lambda_n)}{u'(\lambda_n)^2}.\tag{47}$$

59. We proceed to apply the results in (45) and (47) to the sum (37) which defines the integral (36) whose properties are desired in order to investigate the convergence of (33).

Let us suppose that for the given value of α we can construct a function $\theta(z)$ which shall be analytic throughout the finite z plane and such that its value at the points $\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots$ shall be given by the equation

$$\theta(\lambda_n)=\sum_{s=1}^{m}\frac{H_s(\lambda_n,\alpha)\int_0^t F(\alpha+t)H_s(\lambda_n,\alpha+t)\,dt}{\int_a^b F(t)H_s^2(\lambda_n,t)\,dt}\,u'(\lambda_n).\tag{48}$$

As a result of (45) we shall then have

$$\int_0^t \varphi(n,\alpha,t)\,dt=\frac{1}{2\pi i}\int_{C_n}\frac{\theta(z)}{u(z)}\,dz.\tag{49}$$

If, again, we can construct a function $\theta(z)$ analytic throughout the finite plane and subject to the single restriction

$$\theta'(\lambda_n)=\sum_{s=1}^{m}\frac{H_s(\lambda_n,\alpha)\int_0^t F(\alpha+t)H_s(\lambda_n,\alpha+t)\,dt}{\psi(\lambda_n)\int_a^b F(t)H_s^2(\lambda_n,t)\,dt}\,u'(\lambda_n)^2,\tag{50}$$

where $\psi(z)$ is any function of z analytic throughout the finite z plane and such that $\psi'(\lambda_n)u'(\lambda_n)=\psi(\lambda_n)u''(\lambda_n)$ then, upon applying (47) we shall have

$$\int_0^t \varphi(n,\alpha,t)\,dt=\frac{1}{2\pi i}\int_{C_n}\frac{\theta(z)\psi(z)}{u^2(z)}\,dz.\tag{51}$$

[16] Cf. Chapter I, formula (30).

It thus appears that by means of (49) and (51)[17] the discussion of (37) may sometimes be transferred to that of an integral of a complex variable z. This will be the case in the special developments to be considered in what follows.

60. We now proceed to examine the series (33) in some of its more important cases—viz., those related to the general problem of STURM.[18] Here we have $m = 1$ and, representing by $H(\lambda_n, x)$ the single function $H_1(\lambda_n, x)$, we have by hypothesis

(52) $$\int_a^b F(x)H(\lambda_m, x)H(\lambda_n, x)dx = 0 \quad when \quad n \neq m.$$

Moreover, when x is taken between a and b (a and b included) the function $H(z, x)$ is assumed to be analytic in z throughout the finite z plane and real when z is real; also to be such that when z has any one of the values $\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots$ it is a solution of a certain linear differential equation of the form

(53) $$\frac{\partial}{\partial x}\left(K(x)\frac{\partial H(z, x)}{\partial x}\right) + \{F(x)\nu(z) + F_1(x)\}H(z, x) = 0,$$

where $K(x)$, $F(x)$ and $F_1(x)$ are functions of x only, while $\nu(z)$ is a function of z only.

In such cases the developments (33) assume the form

(54) $$\sum_{n=1}^{\infty} q_n H(\lambda_n, x),$$

where

(55) $$q_n = \frac{\int_a^b f(x)F(x)H(\lambda_n, x)dx}{\int_a^b F(x)H^2(\lambda_n, x)dx}.$$

We first proceed to note certain general consequences which flow from the above restrictions upon $H(z, x)$.

From (53) we have

(56) $$\frac{\partial}{\partial x}\left(K(x)\frac{\partial H(\lambda_m, x)}{\partial x}\right) + \{F(x)\nu(\lambda_m) + F_1(x)\}H(\lambda_m, x) = 0,$$

[17] It is to be observed that if in (49) the function $\theta(z)$ has singular points within C_n the formula continues true provided that the sum of the residues of the right integrand at such points be subtracted from the second member. Similar remarks evidently apply in (51) if $\theta(z)\psi(z)$ has singular points within C_n.

[18] Cf. DINI, "Serie di Fourier, etc.," §§ 90–96. The problem here presented has been the subject of numerous and extensive researches in recent years, but usually under the assumption (not here introduced) that the differential equation (53) in terms of whose solutions the proposed development is to be made, shall have no singular points within the (closed) interval (a, b) for which the same development is to hold. But this assumption unfortunately rules out some of the most important special developments, such as those in terms of Bessel functions and Legendre functions. For summary remarks upon the more recent researches of this character, see BÔCHER'S address before the International Congress of Mathematicians at Cambridge in August, 1912, § 11.

$$(57)\qquad \frac{\partial}{\partial x}\left(K(x)\frac{\partial H(\lambda_n, x)}{\partial x}\right) + \{F(x)\nu(\lambda_n) + F_1(x)\}H(\lambda_n, x) = 0.$$

Hence, after multiplying both members of (56) by $H(\lambda_n, x)$ and both members of (57) by $H(\lambda_m, x)$ and subtracting, we obtain

$$(58)\qquad \begin{aligned} &F(x)\{\nu(\lambda_n) - \nu(\lambda_m)\}H(\lambda_m, x)H(\lambda_n, x) \\ &= \frac{\partial}{\partial x}\left\{K(x)\left[H(\lambda_n, x)\frac{\partial H(\lambda_m, x)}{\partial x} - H(\lambda_m, x)\frac{\partial H(\lambda_n, x)}{\partial x}\right]\right\}\end{aligned}$$

and therefore

$$\int_a^b F(x)H(\lambda_m, x)H(\lambda_n, x)dx = \frac{1}{\nu(\lambda_n) - \nu(\lambda_m)}\left\{K(x)\left[H(\lambda_n, x)\frac{\partial H(\lambda_m, x)}{\partial x} - H(\lambda_m, x)\frac{\partial H(\lambda_n, x)}{\partial x}\right]\right\}_a^b.$$

Thus, in order that (52) may be satisfied it suffices that the roots $\lambda_1, \lambda_2, \lambda_3, \cdots$ be so chosen that

$$(59)\qquad \begin{aligned} &K(b)\left[H(\lambda_n, x)\frac{\partial H(\lambda_m, x)}{\partial x} - H(\lambda_m, x)\frac{\partial H(\lambda_n, x)}{\partial x}\right]_b \\ &- K(a)\left[H(\lambda_n, x)\frac{\partial H(\lambda_m, x)}{\partial x} - H(\lambda_m, x)\frac{\partial H(\lambda_n, x)}{\partial x}\right]_a = 0,\end{aligned}$$

provided $m \neq n$. Moreover, among the different ways in which this relation may exist is that of supposing that for every value of n we have the following two equations simultaneously:

$$(60)\qquad \begin{cases} K(x)\dfrac{\partial H(\lambda_n, x)}{\partial x} - h'H(\lambda_n, x) = 0 \quad \textit{when} \quad x = a, \\[2ex] K(x)\dfrac{\partial H(\lambda_n, x)}{\partial x} - hH(\lambda_n, x) = 0 \quad \textit{when} \quad x = b, \end{cases}$$

h and h' being any real constants, including the limiting values $h = \pm\infty$, $h' = \pm\infty$ corresponding to which the same equations become $H(\lambda_n, a) = 0$ and $H(\lambda_n, b) = 0$ respectively. We shall hereafter confine our attention to the cases in which relations (60) are satisfied. Furthermore, if $K(a) \neq 0$, $K(b) \neq 0$, we shall suppose that the transcendental equation $u(z) = 0$ whose roots determine the quantities $\lambda_1, \lambda_2, \lambda_3, \cdots$ is taken in the one or the other of the two following manners:

$$(61)\qquad u(z) = \left[K(x)\frac{\partial H(z, x)}{\partial x} - h'H(z, x)\right]_a = 0,$$

$$(62)\qquad u(z) = \left[K(x)\frac{\partial H(z, x)}{\partial x} - hH(z, x)\right]_b = 0,$$

thus rendering one of the two relations (60) satisfied at once. Similarly, if $K(a) = 0$, $K(b) \neq 0$ we shall use (62). In this case it is to be observed that we have merely to place $h' = 0$ ($u(z)$ having been chosen as indicated) to have equations (60) satisfied *whatever* the solution $H(z, x)$ of (53) chosen to be used in (54). Likewise, when $K(a) \neq 0$, $K(b) = 0$ we shall use (61) in which case the solution $H(z, x)$ of (53) to be used in (54) may be chosen arbitrarily. Finally, if $K(a) = 0$, $K(b) = 0$ the equation $u(z) = 0$ may be taken arbitrarily together with the solution $H(z, x)$ without destroying the coexistence of (60).

61. We add that if the solution $H(z, x)$ considered as a function of the two variables z and x is finite and continuous together with its first and second partial derivatives: $\partial H/\partial x$, $\partial H/\partial z$, $\partial^2 H/\partial x \partial z$ for all real values of x such that $a \leqq x \leqq b$ and for complex values of z in the neighborhood of each of the points $z = \lambda_n$, and if the equation (61) (h' finite or infinite) is satisfied *identically* for all values of z in these regions, then it is easy to evaluate each of the integrals

$$\int_a^b F(x) H^2(\lambda_n, x) dx, \tag{63}$$

which appear in the coefficients q_n of the series (54).

In fact, if we change λ_m to z, as we may now do, and integrate from a to x ($a < x < b$) we shall have by (58) and (61)

$$\int_a^x F(x) H(z, x) H(\lambda_n, x) dx = \frac{1}{\nu(z) - \nu(\lambda_n)} \left[K(x) \left\{ H(z, x) \frac{\partial H(\lambda_n, x)}{\partial x} - H(\lambda_n, x) \frac{\partial H(z, x)}{\partial x} \right\} \right],$$

and this holds true for any value of z in the indicated regions.

Whence, upon allowing z to approach the value λ_n we obtain under the present hypotheses

$$\int_a^x F(x) H^2(\lambda_n, x) dx = \frac{1}{\nu'(\lambda_n)} \left[K(x) \left\{ \frac{\partial H(\lambda_n, x)}{\partial \lambda_n} \frac{\partial H(\lambda_n, x)}{\partial x} - H(\lambda_n, x) \frac{\partial^2 H(\lambda_n, x)}{\partial \lambda_n \partial x} \right\} \right],$$

where if desired λ_n may be changed to z for values of z in the indicated regions. Passing now to the limit as $x = b$ we obtain

$$\int_a^b F(x) H^2(\lambda_n, x) dx = \frac{1}{\nu'(\lambda_n)} \left[K(x) \left\{ \frac{\partial H(\lambda_n, x)}{\partial \lambda_n} \frac{\partial H(\lambda_n, x)}{\partial x} - H(\lambda_n, x) \frac{\partial^2 H(\lambda_n, x)}{\partial \lambda_n \partial x} \right\} \right]_b, \tag{64}$$

in which as above we may replace λ_n by z provided z has values in the indicated regions.

Finally, by use of the second of equations (60) we may write (64) in the following form when h is finite:

$$(65)\qquad \int_a^b F(x)H^2(\lambda_n, x)dx = \frac{1}{\nu'(\lambda_n)}\left[H(\lambda_n, x)\left\{h\frac{\partial H(\lambda_n, x)}{\partial \lambda_n} - K(x)\frac{\partial^2 H(\lambda_n, x)}{\partial \lambda_n \partial x}\right\}\right]_b.$$

In like manner, if $h = \pm\infty$ so that $H(\lambda_n, b) = 0$ then (64) may be written

$$(66)\qquad \int_a^b F(x)H^2(\lambda_n, x)dx = \frac{1}{\nu'(\lambda_n)}\left[K(x)\frac{\partial H(\lambda_n, x)}{\partial \lambda_n}\frac{\partial H(\lambda_n, x)}{\partial x}\right]_b.$$

62. Expressions (64), (65) and (66) thus enable us to find under special conditions the value of the integral (63). Among the cases in which the same special conditions cannot be satisfied, the following are to be especially noted.

If, as we have supposed, $F(x)$ and $H(\lambda_n, x)$ are real when x is such that $a \leqq x \leqq b$ and if in this interval $F(x)$ does not change sign, then the integral (63) cannot be equal to zero. Whence, under these conditions (64) cannot be used if $K(b) = 0$ (h finite or infinite) or if

$$(67)\qquad H(\lambda_n, b) = 0 \qquad (h \textit{ finite})$$

or if

$$(68)\qquad \left[\frac{\partial H(\lambda_n, x)}{\partial \lambda_n}\right]_b = 0 \quad \textit{or} \quad \left[\frac{\partial H(\lambda_n, x)}{\partial x}\right]_b = 0 \qquad (h \textit{ infinite})$$

or (as appears from (65)) if

$$(69)\qquad \left[h\frac{\partial H(\lambda_n, x)}{\partial \lambda_n} - K(x)\frac{\partial^2 H(\lambda_n, x)}{\partial \lambda_n \partial x}\right]_b = 0 \qquad (h \textit{ finite}).$$

63. Returning then to the series (54) and assuming that the quantities $\lambda_1, \lambda_2, \lambda_3, \cdots$ are taken as the positive roots of the equation (62) while the equation (61) shall be satisfied *identically* for all values (real or complex) of z in the neighborhoods of the same values; assuming also that the partial derivatives of $H(z, x)$ exist and satisfy such other conditions as we have imposed in § 61, we may say that unless $K(b) = 0$ or one of the conditions (67), (68) or (69) is satisfied, we shall have for such developments when h is finite

$$u(z) = \left[K(x)\frac{\partial H(z, x)}{\partial x} - hH(z, x)\right]_b,$$

$$(70)\qquad \int_a^b F(x)H^2(\lambda_n, x)dx = \frac{1}{\nu'(\lambda_n)}\left[H(\lambda_n, x)\left\{h\frac{\partial H(\lambda_n, x)}{\partial \lambda_n} - K(x)\frac{\partial^2 H(\lambda_n, x)}{\partial \lambda_n \partial x}\right\}\right]_b = -\frac{u'(\lambda_n)}{\nu'(\lambda_n)}H(\lambda_n, b).$$

On the other hand, if $h = \pm\infty$, we shall have

$$u(z) = H(z, b),$$

$$(71) \qquad \int_a^b F(x)H^2(\lambda_n, x)dx = \frac{u'(\lambda_n)}{\nu'(\lambda_n)}\left[K(x)\frac{\partial H(\lambda_n, x)}{\partial x}\right]_b.$$

Upon applying formulas (49) and (51) we thus obtain the following general results concerning the integrals (36) pertaining to the present developments:

(1) *h finite.* Formula (49) here gives

$$(72) \qquad \int_0^t \varphi(n, \alpha, t)dt = \frac{1}{2\pi i}\int_{C_n} - \frac{\nu(z)H(z, \alpha)\int_0^t F(\alpha + t)H(z, \alpha + t)dt}{H(z, b)\left[K\frac{\partial H}{\partial x} - hH\right]_b} - R_n,$$

where R_n represents the sum of the residues of the integrand at any singular points which it may have within C_n besides the points $\lambda_1, \lambda_2, \cdots, \lambda_n$; *i. e.*, besides those points $z = \lambda_n$ within C_n, for which

$$(73) \qquad \left[K(x)\frac{\partial H}{\partial x} - hH\right]_b \equiv u(z) = 0.$$

Formula (51) here gives

$$(74) \qquad \int_0^t \varphi(n, \alpha, t)dt = \frac{1}{2\pi i}\int_{C_n} \frac{\theta(z)\psi(z)dz}{\left[K\frac{\partial H}{\partial x} - hH\right]_b^2} - R_n,$$

where R_n represents the sum of the residues of the integrand at any singular points which it may have within C_n besides those points $z = \lambda_n$ for which (73) exists, where $\psi(z)$ is a function of z only such that $\psi'(\lambda_n)u'(\lambda_n) - \psi(\lambda_n)u''(\lambda_n) = 0$ and where $\theta(z)$ is to be so determined that

$$(75) \qquad \theta'(\lambda_n) = - \frac{\nu'(\lambda_n)u'(\lambda_n)H(\lambda_n, \alpha)\int_0^t F(\alpha + t)H(\lambda_n, \alpha + t)dt}{\psi(\lambda_n)H(\lambda_n, b)}$$

(2) $h = \pm\infty$. Formula (49) here gives

$$(76) \qquad \int_0^t \varphi(n, \alpha, t)dt = \frac{1}{2\pi i}\int_{C_n} \frac{\nu'(z)H(z, \alpha)\int_0^t F(\alpha + t)H(z, \alpha + t)dt}{\left(K\frac{\partial H}{\partial x}\right)_b H(z, b)} dz - R_n,$$

where R_n represents the sum of the residues of the integrand at any singular points which it may have within C_n besides those points for which

$$H(z, b) \equiv u(z) = 0.$$

Formula (51) here gives

$$(77) \qquad \int_0^t \varphi(n, \alpha, t)dt = \frac{1}{2\pi i}\int_{C_n} \frac{\varphi(z)\theta(z)dz}{H^2(z, b)} - R_n,$$

where R_n represents the sum of the residues of the integrand at a
points which it may have within C_n besides those points for which

$$H(z,\, b) \equiv u(z) = 0,$$

where

$$\psi'(\lambda_n)u'(\lambda_n) - \psi(\lambda_n)u''(\lambda_n) = 0$$

and where

(78) $$\theta'(\lambda_n) = \frac{\nu'(\lambda_n)u'(\lambda_n)H(\lambda_n,\, \alpha)\int_0^{c} F(\alpha + t)H(\lambda_n,\, \alpha + t)dt}{\psi(\lambda_n)\left(K\dfrac{\partial H}{\partial x}\right)_b}.$$

IV

The Summability and Convergence of Important Special Developments. Developments in Terms of Bessel Functions, Legendre Functions, etc.

1. *Certain Important Sine Developments.*

64. As the simplest application of the preceding general results to well-known developments in mathematical physics, we now turn to the development

$$\sum_{n=1}^{\infty} q_n \sin \lambda_n x, \tag{79}$$

where

$$q_n = 2\frac{\lambda_n^2 + p^2}{\lambda_n^2 + p(p+1)}\int_0^1 f(x) \sin \lambda_n x \, dx \tag{80}$$

and where the quantities λ_n are the positive roots of the equation

$$z \cos z + p \sin z = 0, \tag{81}$$

p being a (real) constant $\neq -1$.[19]

In this case $H(z, x) = \sin zx$ so that the differential equation (53) becomes

$$\frac{\partial^2 H(z, x)}{\partial x^2} + z^2 H(z, x) = 0. \tag{82}$$

Thus, we have $K(x) = 1$, $F(x) = 1$, $F_1(x) = 0$ and, as appears from (80), $a = 0$, $b = 1$.

Moreover, equation (61) becomes satisfied *identically* for all values of z if we place $h' = \infty$, while equation (62) becomes (81) if we place $h = -p$.

Considering then that, in the notation of § 60, we here have

$$u(z) = z \sin z + p \sin z$$

and noting that the solution $\sin zx$ of (82) is one to which the general results obtained in §§ 61, 63 apply, we may write by use of (64) and (81),

$$\int_0^1 \sin^2 zx \, dx = \frac{1}{2z}\left[\frac{\partial \sin zx}{\partial x}\frac{\partial \sin zx}{\partial z} - \sin zx \frac{\partial^2 \sin zx}{\partial z \partial x}\right]_{x=1}$$

$$= \frac{1}{2z}[z \cos^2 z - \sin z \cos z + z \sin^2 z] = \frac{\sin^2 z}{2z^2}[z^2 + p(p+1)]$$

and hence

$$\int_0^1 \sin^2 \lambda_n x\,dx = \frac{\sin^2 \lambda_n}{2\lambda_n^2}[\lambda_n^2 + p(p+1)]$$

or, since

$$\sin^2 \lambda_n = \frac{\lambda_n^2}{\lambda_n^2 + p^2}$$

we may write

$$\int_0^1 \sin^2 \lambda_n x\,dx = \frac{\lambda_n^2 + p(p+1)}{2(\lambda_n^2 + p^2)}.$$

Thus it appears in the first place that the coefficients q_n as calculated by (55) agree with the given values (80).

Now,

$$u'(z) = -z\sin z + \cos z + p\cos z = -z\sin z + (1+p)\frac{u(z) - p\sin z}{z},$$

$$u''(z) = -\sin z - z\cos z - \sin z - p\sin z = -[2\sin z + u(z)],$$

and hence

$$(83)\qquad \begin{cases} u'(\lambda_n) = -\dfrac{\sin \lambda_n}{\lambda_n}[\lambda_n^2 + p(p+1)], \\ u''(\lambda_n) = -2\sin \lambda_n, \end{cases}$$

so that

$$(84)\qquad \int_0^1 \sin^2 \lambda_n x\,dx = \frac{1}{2}\frac{u'(\lambda_n)^2}{\lambda_n^2 + p(p+1)}.$$

Let us now avail ourselves of formula (77).[20] In order to do this we are first to determine the function $\psi(z)$ according to the condition

$$\psi'(\lambda_n)u'(\lambda_n) - \psi(\lambda_n)u''(\lambda_n) = 0.$$

A possible choice of $\psi(z)$ is $\psi(z) = z^2 + p(p+1)$ since from (83) we have

$$\frac{u''(\lambda_n)}{u'(\lambda_n)} = \frac{2\lambda_n}{\lambda_n^2 + p(p+1)}.$$

Assuming that $\psi(z)$ has been chosen in this manner, we now have to determine a function $\theta(z)$ according to the condition (78) which, by means of (84) becomes in the present instance

$$\theta'(\lambda_n) = 2[\lambda_n^2 + p(p+1)]\frac{\sin \lambda_n \alpha \displaystyle\int_0^t \sin \lambda_n(\alpha + t)\,dt}{\psi(\lambda_n)}$$

$$= 2\sin \lambda_n \alpha \int_0^t \sin \lambda_n(\alpha + t)\,dt.$$

[20] Dini has shown through an elaborate investigation that this formula will always lead to decisive results whenever the solution $H(z, x)$ has the special form $H(zx)$; that is, when the variables z and x enter only through their product. (Cf. "Serie di Fourier, etc.," §§ 97–109.) The well-known developments in terms of Bessel functions form a special class of this kind.

Hence, let us take $\theta(z)$ such that

$$\theta'(z) = 2 \sin z\alpha \int_0^t \sin z(\alpha + t)dt = \int_0^t [\cos tz - \cos (2\alpha + t)]dt.$$

In particular, let us take

$$\theta(z) = \int_0^t \int_0^z [\cos tz - \cos (2\alpha + t)z]dzdt = \int_0^t \left[\frac{\sin tz}{t} - \frac{\sin (2\alpha + t)z}{2\alpha + t}\right] dt.$$

Formula (77) thus becomes

$$\int_0^t \varphi(n, \alpha, t)dt = \frac{1}{2\pi i}\int_{C_n} \frac{z^2 + p(p+1)}{[z \cos z + p \sin z]^2}\int_0^t \left[\frac{\sin tz}{t} - \frac{\sin (2\alpha + t)z}{2\alpha + t}\right] dtdz$$

(85)
$$= \frac{1}{2\pi i}\int_0^t dt \int_{C_n} \frac{1 + \omega_1(z)}{[\cos z + \omega_2(z) \sin z]^2}\frac{\sin tz}{t} dz$$

$$- \frac{1}{2\pi i}\int_0^t \frac{dt}{2\alpha + t}\int_{C_n} \frac{[1 + \omega_1(z)] \sin (2\alpha + t)z}{[\cos z + \omega_2(z) \sin z]^2} dz,$$

where the contour C_n is so taken as to inclose the roots $\lambda_1, \lambda_2, \cdots, \lambda_n$ and only these roots of the equation $u(z) = 0$ and where, in the last two integrals we have placed for simplicity

$$\omega_1(z) = \frac{p(p+1)}{z^2}, \qquad \omega_2(z) = \frac{p}{z}.$$

We observe at this point that in applying Theorems I and II of §§ 51, 52 to the function $\varphi(n, \alpha, t)$ of the present development, the values of α and t with which we shall be concerned are such that

(86)
$$\begin{cases} 0 < a' < \alpha < b' < 1, \\ - \alpha \leqq t \leqq 1 - \alpha, \\ 0 < a' < \alpha \leqq 2\alpha + t \leqq 1 + \alpha < 1 + b' < 2. \end{cases}$$

Returning then to (85), let us take as the contour C_n the rectangle formed in the z plane ($z = x + iy$) by the lines $z = x + ij$, $z = x - ij$, $z = iy$, $z = k + iy$; j being any positive quantity arbitrarily large and k being any positive quantity lying between λ_n and λ_{n+1}. Now, the function appearing in the integrand of (85) is an odd function of z which remains finite in the neighborhood of the point $z = 0$ since $p \neq -1$. Whence, the portion of the integral in question due to integration over the y-axis is equal to zero. Upon the sides which are parallel to the x-axis we have $dz = dx$. Whence, considering first the side upon which $z = x + ij$ the last integral of (85) extended over this side becomes

$$-\frac{1}{2\pi i}\int_0^t \frac{dt}{A}\int_0^k \frac{\{1 + \omega_1\} \{\sin Ax \cosh Aj + i \cos Ax \sinh Aj\}}{D_1 D_2} dx,$$

where $A = 2\alpha + t$, $D_1 = \cos x \cosh j - i \sin x \sinh j + \omega_2$, $D_2 = \sin x \cosh j + i \cos x \sinh j$.

Now, the functions $\omega_1 = \omega_1(z)$, $\omega_2 = \omega_2(z)$ are each less in absolute value than a constant (independent of z) provided $|z| > Q = \textit{fixed number} > 0$. Thus, we have but to make use of the well-known properties of the hyperbolic functions to see that if we place $j = +\infty$ the expression above will approach *uniformly* the limit zero for all α and t satisfying relations (86).

Similarly, we reach the same result for the last integral of (85) when extended over the side upon which $z = x - ij$.

Turning now to the first integral in the second member of (85) extended over the sides upon which $z = x \pm ij$, we note that

$$\frac{\sin tz}{t} = \frac{\sin tx}{t}\cosh tj \pm i \cos tx \frac{\sinh tj}{t}$$

and hence,

$$\frac{\sin tz}{t} = x \cos t_1 x \cosh tj \pm ij \cos tx \sinh t_2 j,$$

where t_1 and t_2 are values lying between 0 and t. Moreover, for all values of t under consideration in (86) we have $|t| < 1$ so that if we place $j = +\infty$ as before, the first integral in the second member of (85), like its last integral, will approach uniformly the limit zero for all values of α and t concerned in (86).

We turn then to the consideration of the last member of (85) when extended over the side of the rectangle C_n which is parallel to the y-axis. Here we have $z = k + iy$, $dz = i dy$ and, having taken $j = +\infty$, we see from what has just been said that for all values of α and t in (86) this member reduces to

$$(87)\qquad \frac{1}{2\pi}\int_0^t dt \int_{-\infty}^{\infty} \frac{1 + \omega_1}{[\cos z + \omega_2 \sin z]^2}\frac{\sin tz}{t}dy \\ - \frac{1}{2\pi}\int_0^t \frac{dt}{2\alpha + t}\int_{-\infty}^{\infty} \frac{\{1 + \omega_1\} \sin (2\alpha + t)z}{[\cos z + \omega_2 \sin z]^2}dy,$$

in which it is understood that $z = k + iy$.

Now, it suffices for our purpose to examine the behavior of (87) as $k = \infty$ and we may take for k any number which, at least for all values of n greater than some fixed value, increases indefinitely with n without at any time being a root of the equation $u(z) = 0$; *i. e.*, of the equation

$$E = \cos z + \omega_2 \sin z = 0.$$

Thus, we may take $k = n\pi$, in which case

$$E^2 = [\cos (n\pi + iy) + \omega_2 \sin (n\pi + iy)]^2 = \cosh^2 y[1 + i\omega_2 \tanh y]^2$$

and hence

$$(88)\qquad \begin{aligned}\frac{1}{E^2} &= \frac{1}{\cosh^2 y}\left\{1 - 2i\omega_2 \tanh y - \omega_2^2 \frac{3\tanh^2 y + 2i\omega_2 \tanh^3 y}{1 + 2i\omega_2 \tanh y - \omega_2^2 \tanh^2 y}\right\}\\ &= \frac{1}{\cosh^2 y}\left\{1 + \gamma\frac{\tanh y}{z} + \frac{\delta}{z^2}\right\},\end{aligned}$$

where, upon recalling the form of $\omega_2(z)$, we have $\gamma = -2ip$ and therefore independent of z, while δ depends upon z but has a modulus less than a certain fixed number M for all values of $|z| >$ a fixed number k_0.

Thus expression (87) becomes

$$(89)\qquad \begin{aligned}&\frac{1}{2\pi}\int_0^t dt \int_{-\infty}^{\infty}\left(1 + \frac{\gamma}{z}\tanh y + \frac{\delta_1}{z^2}\right)\frac{\sin tz}{t\cosh^2 y}dy\\ &\qquad - \frac{1}{2\pi}\int_0^t \frac{dt}{2\alpha + t}\int_{-\infty}^{\infty}\left(1 + \frac{\gamma}{z}\tanh y + \frac{\delta_1}{z^2}\right)\frac{\sin(2\alpha + t)z}{\cosh^2 y}dy,\end{aligned}$$

where

$$1 + \frac{\gamma}{z}\tanh y + \frac{\delta_1}{z^2} = [1 + \omega_1]\left[1 + \frac{\gamma}{z}\tanh y + \frac{\delta}{z^2}\right],$$

so that δ_1 like δ has a modulus less than some constant M_1 when $|z| >$ *a fixed number* $= k_1$.

Considering now the terms in (89) which have z^2 in their denominator, we see that for all values of α and t in (86) these terms approach uniformly the limit zero as $k = \infty$. Thus, since

$$(90)\qquad \begin{aligned}\left|\frac{\sin(2\alpha + t)z}{\cosh^2 y}\right| &\leqq \left|\sin(2\alpha + t)k\frac{\cosh(2\alpha + t)y}{\cosh^2 y}\right|\\ &\qquad + \left|\cos(2\alpha + t)k\frac{\sinh(2\alpha + t)y}{\cosh^2 y}\right|,\end{aligned}$$

where $0 < a' \leqq 2\alpha + t \leqq 1 + b' < 2$ we have, however great $|z|$ may be,

$$\left|\frac{\sin(2\alpha + t)z}{\cosh^2 y}\right| < 2,$$

so that

$$\left|\frac{-1}{2\pi}\int_0^t \frac{dt}{2\alpha + t}\int_{-\infty}^{\infty}\frac{\delta_1}{z^2}\frac{\sin(2\alpha + t)z}{\cosh^2 y}dy\right| < \frac{1}{2\pi}\int_0^1\frac{dt}{a'}2M_1\int_0^{\infty}\frac{dy}{k^2 + y^2} = \frac{M_1}{a'k}.$$

In like manner, noting that

$$(91)\qquad \begin{aligned}\frac{\sin tz}{t} &= \frac{\sin tk}{t}\cosh ty + i\cos tk\frac{\sinh ty}{t}\\ &= k\cos t_1 k\cosh ty + iy\cos tk\cosh t_2 k,\end{aligned}$$

where t_1 and t_2 lie between 0 and t, and recalling that for all values of t under

consideration we have $|t| < 1$, we may write

$$\left|\frac{1}{2\pi}\int_0^t dt\int_{-\infty}^{\infty}\frac{\delta_1}{z^2}\frac{\sin tz}{t\cosh^2 y}dy\right| \leqq \frac{M_1}{\pi}\int_0^1 dt\int_{-\infty}^{\infty}\frac{dy}{k^2+y^2}=\frac{M_1}{k}.$$

Similarly it appears that the derivative with respect to t of each of the same terms of (89) has the properties just indicated.

Thus (89) reduces to the form

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^t dt\int_{-\infty}^{\infty}\left(1+\frac{\gamma(k-iy)}{k^2+y^2}\tanh y\right)\frac{\sin kt\cosh ty}{t\cosh^2 y}dy\\
&+\frac{i}{2\pi}\int_0^t dt\int_{-\infty}^{\infty}\left(1+\frac{\gamma(k-iy)}{k^2+y^2}\tanh y\right)\frac{\cos kt\sinh ty}{t\cosh^2 y}dy\\
(92)\quad &-\frac{1}{2\pi}\int_0^t\frac{dt}{2\alpha+t}\int_{-\infty}^{\infty}\left(1+\frac{\gamma(k-iy)}{k^2+y^2}\tanh y\right)\frac{\sin(2\alpha+t)k\cosh(2\alpha+t)y}{\cosh^2 y}dy\\
&-\frac{i}{2\pi}\int_0^t\frac{dt}{2\alpha+t}\int_{-\infty}^{\infty}\left(1+\frac{\gamma(k-iy)}{k^2+y^2}\tanh y\right)\frac{\cos(2\alpha+t)k\sinh(2\alpha+t)y}{\cosh^2 y}dy\\
&+\Lambda(\alpha,t,k),
\end{aligned}$$

where, for all values of α and t under consideration, $\Lambda(\alpha, t, k)$ and $d\Lambda(\alpha, t, k)/dt$ converge uniformly to zero when $k = \infty$. Or, expanding and dropping integrals which vanish identically since they are relative to odd functions of y, (92) assumes the form

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^t\frac{\sin kt}{t}dt\int_{-\infty}^{\infty}\frac{\cosh ty}{\cosh^2 y}dy-\frac{\gamma i}{2\pi}\int_0^t\frac{\sin kt}{t}dt\int_{-\infty}^{\infty}\frac{y\tanh y\cosh ty}{(k^2+y^2)\cosh^2 y}dy\\
&+\frac{\gamma i}{2\pi}\int_0^t\cos kt\,dt\int_{-\infty}^{\infty}\frac{k\tanh y\sinh ty}{(k^2+y^2)t\cosh^2 y}dy\\
(93)\quad &-\frac{1}{2\pi}\int_0^t\frac{\sin(2\alpha+t)k}{2\alpha+t}dt\int_{-\infty}^{\infty}\frac{\cosh(2\alpha+t)y}{\cosh^2 y}dy\\
&+\frac{\gamma i}{2\pi}\int_0^t\frac{\sin(2\alpha+t)k}{2\alpha+t}dt\int_{-\infty}^{\infty}\frac{y}{k^2+y^2}\frac{\tanh y\cosh(2\alpha+t)y}{\cosh^2 y}dy\\
&-\frac{\gamma i}{2\pi}\int_0^t\frac{\cos(2\alpha+t)k}{2\alpha+t}dt\int_{-\infty}^{\infty}\frac{k}{k^2+y^2}\frac{\tanh y\sinh(2\alpha+t)y}{\cosh^2 y}dy+\Lambda(\alpha,t,k).
\end{aligned}$$

We proceed to consider separately the six integrals here appearing.

The first may be put into the form

$$(94)\qquad I_1=\frac{f_1(0)}{2\pi}\int_0^t\frac{\sin kt}{\sin t}dt+\frac{1}{2\pi}\int_0^t[f_1(t)-f_1(0)]\frac{\sin kt}{\sin t}dt,$$

where

$$f_1(t)=\frac{\sin t}{t}\int_{-\infty}^{\infty}\frac{\cosh ty}{\cosh^2 y}dy.$$

Whence, if $t > 0$ the limit of the first term in the last member as $k = \infty$ is $f_1(0)/4$ (cf. Appendix, Lemma II). But

$$f_1(0) = \int_{-\infty}^{\infty} \frac{dy}{\cosh^2 y} = 2,$$

and hence as $k = \infty$ the term just mentioned approaches the limit $\frac{1}{2}$ when $t > 0$. Again, by breaking up the integration in the last term of (94) into that from $t = 0$ to $t = \eta$ plus that from $t = \eta$ to $t = t$ (η arbitrarily small and > 0) and observing that the function $f_1(t)$ has limited total fluctuation in the neighborhood of the point $t = 0$, it follows that the same term approaches the limit zero as $k = \infty$ (see Appendix, Lemmas I, III).

Likewise, if $t < 0$ we obtain $\lim\limits_{k=\infty} I_1 = -\frac{1}{2}$.

The second and third integrals of (93) may be reduced respectively to the forms

$$-\frac{\gamma i}{2\pi k} \int_0^t \frac{\sin kt}{kt}\, dt \int_{-\infty}^{\infty} \frac{k^2}{k^2 + y^2} \frac{y \tanh y \cosh ty}{\cosh^2 y}\, dy,$$

$$\frac{\gamma i}{2\pi k} \int_0^t \cos kt\, dt \int_{-\infty}^{\infty} \frac{k^2}{k^2 + y^2} \frac{y \tanh y \cosh t_1 y}{\cosh^2 y}\, dy,$$

where t_1 is a quantity lying between 0 and t. Since we have always

$$\frac{k^2}{k^2 + y^2} \leqq 1, \qquad \frac{\sin kt}{kt} < 1,$$

it thus appears that the limit approached by each of these integrals as $k = \infty$ is equal to zero.

In order to study the fourth integral of (93) let us make therein the substitution $2\alpha + t = 2 - \tau$. Since $k = n\pi$ the integral in question becomes

(95) $$-\frac{1}{2\pi} \int_{2(1-\alpha)}^{\tau} \frac{\sin k\tau}{2 - \tau}\, d\tau \int_{-\infty}^{\infty} \frac{\cosh (2 - \tau)y}{\cosh^2 y}\, dy,$$

in which it is to be noted that for all values of α and t in (86) the quantity τ is positive ($1 - b' < \tau < 2 - a'$).

The expression (95) is of the form

(96) $$-\frac{1}{2\pi} \int_{2(1-\alpha)}^{\tau} f_1(\tau) \frac{\sin k\tau}{\sin \tau}\, d\tau,$$

where

(97) $$f_1(\tau) = \frac{\sin \tau}{2 - \tau} \int_{-\infty}^{\infty} \frac{\cos (2 - \tau)y}{\cosh^2 y}\, dy.$$

We have now but to apply Lemma I of the Appendix to the integral (96) in order to see that for all values of α and t with which we are concerned the expression (96) converges uniformly to zero when $k = \infty$.

Finally, the fifth and sixth integrals of (93) are readily seen to approach the limit zero when $k = \infty$ and the convergence is uniform for all values of α and t entering into (86), since we have always $|2\alpha + t| < 2$.

In summary, then, the present function $\varphi(n, \alpha, t)$ satisfies relation (I) of Theorem I, § 51, it being understood that we here have $a = 0$, $b = 1$.

We turn therefore to a consideration of relation (II) of the same Theorem. This relation is at once seen to be satisfied since, as just shown, all the integrals of (93) converge uniformly to zero for all α and t under consideration except the first, and this integral satisfies relation (II) of the theorem by virtue of Lemma III of the Appendix.

Again, relation (III) of Theorem I, § 51 is readily seen to be satisfied in the present instance upon noting that the function $\varphi(n, \alpha, t)$ is here equal to the derivative with respect to t of the expression (93) and that $d\Lambda(\alpha, t, k)/dt$ converges uniformly to zero, as already pointed out, when $k = \infty$.

Before summarizing these results into a theorem respecting the series (79) we turn to consider the application which may be made in the present instance of the general Theorem II of § 52, thus arriving at certain results concerning the summability of the same series. In view of the existence already demonstrated of relations (I) and (III) of § 51 it will here suffice to consider whether relation (II)′ of § 52 is here fulfilled. Moreover, the properties of the integral

$$(98) \qquad \int_0^t |\Phi(n, \alpha, t)|\, dt; \qquad -\epsilon \leqq t \leqq \epsilon$$

of the present development are readily obtained from the expression (93). In fact, in order to be assured of the desired properties of (98), it suffices to show that each of the seven terms of (93) when affected by the operation

$$\frac{1}{n}\sum_{n=0}^{n}$$

has these properties, it being understood that absolute values are employed under each integral sign and in the integrals which constitute the expression $\Lambda(\alpha, t, k)$. For the sake of simplicity and also because the indicated studies are readily carried out, though the forms in (93) are complicated, we shall here suppress the details, noting simply that the desired result follows in each case when we make use of Lemmas IV and V of the Appendix and make use also of (90) and (91) in the study of

$$\frac{1}{n}\sum_{n=0}^{n}\Lambda(\alpha, t, k).$$

We turn then to note the application of Theorem VI of § 55 to the present development in order to ascertain the limit approached by the series (79) when $x = 1$.

For this purpose we first observe that the integral

$$\int_0^t \varphi(n, 1, t)dt$$

is here obtained by placing $\alpha = 1$ in the expression (93). In the resulting new expression the first three integrals, when considered for values of t such that $-1 \leqq t \leqq -\epsilon$ are readily seen to have the properties already obtained for the corresponding integrals for the case $0 < \alpha < 1$ (in which case $-1 < t$ instead of $-1 \leqq t$).

The fourth term, however, does not approach the limit zero in case $\alpha = 1$ since the lower limit of integration in (95) is now equal to zero so that the reasoning before employed can not be used. The resulting integral now assumes the character of the first integral of (93) and if treated in the manner naturally suggested by the analysis of that integral we find directly that for the values of τ under consideration the limit approached as $k = \infty$ is $-f_1(0)/4$ where $f_1(0)$ is to be determined from (97). In order to find the value of $f_1(0)$ it is desirable to make first the following general observation:

If $\varphi(y)$ is a function of the real variable y which, together with its first derivative, is finite for all values of y then, for any number θ such that $|\theta| < 2$ we shall have

$$\int_{-\infty}^{\infty} \varphi(y)\frac{\cosh \theta y}{\cosh^2 y}dy = \frac{\theta}{4-\theta^2}\int_{-\infty}^{\infty}\varphi'(y)\frac{\sinh \theta y}{\cosh^2 y}dy$$
(99)
$$+\frac{2}{4-\theta^2}\int_{-\infty}^{\infty}\varphi'(y)\frac{\cosh \theta y \sinh y}{\cosh^3 y}dy + \frac{6}{4-\theta^2}\int_{-\infty}^{\infty}\varphi(y)\frac{\cosh \theta y}{\cosh^4 y}dy.$$

In fact, integrating once by parts we obtain

$$\int_{-\infty}^{\infty}\varphi(y)\frac{\cosh \theta y}{\cosh^2 y}dy = -\frac{1}{\theta}\int_{-\infty}^{\infty}\varphi'(y)\frac{\sinh \theta y}{\cosh^2 y}dy$$
(100)
$$+\frac{2}{\theta}\int_{-\infty}^{\infty}\varphi(y)\frac{\sinh \theta y \sinh y}{\cosh^3 y}dy$$

and in like manner we may obtain also

$$\int_{-\infty}^{\infty}\varphi(y)\frac{\sinh \theta y \sinh y}{\cosh^3 y}dy = -\frac{1}{\theta}\int_{-\infty}^{\infty}\varphi'(y)\frac{\cosh \theta y \sinh y}{\cosh^3 y}dy$$
(101)
$$+\frac{2}{\theta}\int_{-\infty}^{\infty}\varphi(y)\frac{\cosh \theta y}{\cosh^2 y}dy - \frac{3}{\theta}\int_{-\infty}^{\infty}\varphi'(y)\frac{\cosh \theta y}{\cosh^4 y}dy.$$

Whence, upon combining (100) and (101) we obtain

$$\int_{-\infty}^{\infty}\varphi(y)\frac{\cosh \theta y}{\cosh^2 y}dy = -\frac{1}{\theta}\int_{-\infty}^{\infty}\varphi'(y)\frac{\sinh \theta y}{\cosh^2 y}dy - \frac{2}{\theta^2}\int_{-\infty}^{\infty}\varphi'(y)\frac{\cosh \theta y \sinh y}{\cosh^3 y}dy$$
$$+\frac{4}{\theta^2}\int_{-\infty}^{\infty}\varphi(y)\frac{\cosh \theta y}{\cosh^2 y}dy - \frac{6}{\theta^2}\int_{-\infty}^{\infty}\varphi(y)\frac{\cosh \theta y}{\cosh^4 y}dy$$

and this equation at once gives (99).

Similarly, we may find an analogous form for the integral

$$\int_{-\infty}^{\infty} \varphi(y) \frac{\sinh \theta y}{\cosh^2 y} dy.$$

Thus, for the function $f_1(0)$ where $f_1(\tau)$ is defined by (73) we may write

$$f_1(0) = \lim_{\tau=0} \frac{\sin \tau}{2-\tau} \int_{-\infty}^{\infty} \frac{\cosh (2-\tau)y}{\cosh^2 y} dy = \lim_{\tau=0} \left[\frac{6 \sin \tau}{\tau(4-\tau^2)(2-\tau)} \int_{-\infty}^{\infty} \frac{\cosh (2-\tau)y}{\cosh^4 y} dy \right]$$

$$= \frac{3}{4} \int_{-\infty}^{\infty} \frac{\cosh 2y}{\cosh^4 y} dy = \frac{3}{4} \left(\int_{-\infty}^{\infty} \frac{dy}{\cosh^2 y} + \int_{-\infty}^{\infty} \frac{\tanh^2 y}{\cosh^2 y} dy \right)$$

$$= \frac{3}{4} \left([\tanh y]_{-\infty}^{\infty} + \frac{1}{3} [\tanh^3 y]_{-\infty}^{\infty} \right) = 2.$$

As to the fifth integral of (93) when $\alpha = 1$, the values of t to be considered are as before those for which $0 \leqq t \leqq -1$ and for these we have

$$|2\alpha + t| = |2 + t| \leqq 2$$

instead of $|2 + t| < 2$. The reasoning employed in studying the corresponding integral when $0 < \alpha < 1$ can not therefore be employed. However, if we break up the integral in question into that from $t = 0$ to $t = \epsilon_1$ plus that from $t = \epsilon_1$ to $t = t$ (ϵ_1 arbitrarily small but > 0) the last of the two integrals thus obtained will have the limit zero as $k = \infty$ since for the values of t concerned we have $|2\alpha + t| \leqq 2 - \epsilon_1 < 2$; while the first of the same integrals may be made arbitrarily small with ϵ_1 since by placing

$$\frac{\tanh y \cosh (2+t)y}{\cosh^2 y} = \varphi(y) \frac{\cosh (2+t)y}{\cosh^2 y}; \qquad \varphi(y) = \tanh y$$

and applying (99) we see that the integral

$$\int_{-\infty}^{\infty} \frac{y}{k^2 + y^2} \varphi(y) \frac{\cosh (2+t)y}{\cosh^2 y} = dy$$

remains less in absolute value than a constant independent of ϵ_1 for all values of t such that $-\epsilon_1 \leqq t \leqq 0$.

Similarly, when $\alpha = 1$ the sixth term of (93) may be neglected in the limit as $k = \infty$.

Thus condition $(\mathrm{I})_b$ of Theorem VI, § 55, becomes satisfied in which in the present instance we have $G_2 = -\frac{1}{2} - \frac{1}{2} = -1$ ($\alpha = 1$, $a = 0$, $b = 1$).

Relations $(\mathrm{II})_b$ and $(\mathrm{III})_b$ of § 55 as well as $(\mathrm{II})_b'$ are now readily seen to be satisfied (as in the studies already carried out in connection with (93)) so that by virtue of the general theorems of §§ 51–55 we reach in summary the following

THEOREM. *If $f(x)$ remains finite throughout the interval $(0, 1)$ with the possible exception of a finite number of points and is such that the integral*

$$\int_0^1 |f(x)|\,dx \tag{102}$$

exists, then the series

$$\sum_{n=1}^{\infty} q_n \sin \lambda_n x \tag{103}$$

in which

$$q_n = 2\,\frac{\lambda_n^2 + p^2}{\lambda_n^2 + p(p+1)} \int_0^1 f(x) \sin \lambda_n x\,dx; \qquad p = \text{constant} \neq -1,$$

λ_n being the nth positive root of the equation

$$z \cos z + p \sin z = 0,$$

will converge at any point x $(0 < x < 1)$ in the arbitrarily small neighborhood of which $f(x)$ has limited total fluctuation, and the sum will be

$$\tfrac{1}{2}\,[f(x-0) + f(x+0)].$$

Moreover, the convergence will be uniform to the limit $f(x)$ throughout any interval (a', b') enclosed within a second interval (a_1, b_1) such that $0 < a_1 < a' < b' < b_1 < 1$ provided that $f(x)$ is continuous throughout (a', b') inclusive of the end points $x = a'$, $x = b'$ and has limited total fluctuation throughout (a_1, b_1).

Also, if $f(x)$ remains finite throughout the interval $(0, 1)$ with the possible exception of a finite number of points and is such that the integral (102) exists, then the series (103) will be summable $(r = 1)$ at any point x $(0 < x < 1)$ at which the limits $f(x-0)$, $f(x+0)$ exist and the sum will be

$$\tfrac{1}{2}\,[f(x-0) + f(x+0)].$$

Moreover, the summability will be uniform (§ 45) to the limit $f(x)$ throughout any interval (a', b') such that $0 < a' < b' < 1$ provided that at all points within (a', b'), inclusive of the end points $x = a'$, $x = b'$, the function $f(x)$ is continuous.

Under the same conditions for $f(x)$ when considered throughout the whole interval $(0, 1)$, the series (103), when considered for the value $x = 1$, will converge to the limit $f(1-0)$ provided $f(x)$ is of limited total fluctuation in the neighborhood at the left of the point $x = 1$ and will be summable $(r = 1)$ to the limit $f(1-0)$ whenever this limit exists.

65. It may be observed that in the excluded case for which $p = -1$ the methods which we have followed may be readily altered so as to yield corresponding results. In this case the integrand of (85) has a pole at the point $z = 0$ so that this point should be excluded from the contour C_n. Supposing this to have been accomplished by means of a small semicircle extending to the right of $z = 0$, we may then take as C_n the resulting contour in part rectangular

and in part semicircular. If the integrations be now carried out as before over the respective portions of C_n, that arising from the semicircle will be equal to $-\frac{1}{2}r$ where r represents the residue of the integrand of (85) corresponding to the pole $z=0$. Except for this auxiliary term, the reductions are the same as before, so that in applying the general theorems of §§ 51–55 we encounter an application of the remark (3) in § 56. A similar instance will occur in connection with the developments in terms of Bessel functions, to which we now turn, and in that case we shall elaborate the consequences at some length, though such studies will be omitted for the sake of brevity in connection with the present series (103).

2. *The Developments in Terms of Bessel Functions.*

66. As a second application of the general results obtained in §§ 51–55 we shall now consider certain developments in terms of the function $P_\nu(z)$ defined by the equation

$$P_\nu(z) = \frac{J_\nu(z)}{z^\nu}$$

where $J_\nu(z)$ represents Bessel's function of order ν. The developments in question are closely related to the well known developments for an arbitrary function in terms of Bessel functions and at once yield, as we shall show, results of considerable generality concerning the summability and convergence of the latter.

For the function $P_\nu(z)$ as thus defined, we have, when $\nu \neq$ negative integer,

(104)
$$P_\nu(zx) = \frac{J_\nu(zx)}{(zx)^\nu} = \frac{1}{2^\nu\Gamma(\nu+1)}\left[1 - \frac{(zx)^2}{2^2(\nu+1)} + \frac{(zx)^4}{2^4\cdot 2!\,(\nu+1)(\nu+2)} - \frac{(zx)^6}{2^6\cdot 3!\,(\nu+1)(\nu+2)(\nu+3)} + \cdots\right]$$

while the equation (53) becomes

$$\frac{\partial\left(x^{2\nu+1}\dfrac{\partial P_\nu(zx)}{\partial x}\right)}{\partial x} + z^2x^{2\nu+1}P_\nu(zx) = 0$$

or, placing for brevity $P_\nu(zx) = P$,

(105)
$$x\frac{\partial^2 P}{\partial x^2} + (2\nu+1)\frac{\partial P}{\partial x} + z^2xP = 0.$$

Taking $a=0$, $b=1$, the development (54) in terms of the functions $P_\nu(\lambda_n x)$ becomes

(106)
$$\sum_{n=1}^{\infty} q_n P_\nu(\lambda_n x),$$

where

(107)
$$q_n = \frac{\int_0^1 x^{2\nu+1} f(x) P_\nu(\lambda_n x)\,dx}{\int_0^1 x^{2\nu+1} P_\nu{}^2(\lambda_n x)\,dx}.$$

Equations (60) become

(108)
$$\begin{cases} x^{2\nu+1}\dfrac{\partial}{\partial x}P_\nu(\lambda_n x) - h' P_\nu(\lambda_n x) = 0 & when \quad x = 0, \\ x^{2\nu+1}\dfrac{\partial}{\partial x}P_\nu(\lambda_n x) - h P_\nu(\lambda_n x) = 0 & when \quad x = 1. \end{cases}$$

Of these the first is seen to be satisfied identically for all values of z if we place $h' = 0$ and assume $\nu > -1$, while the second gives as the equation $u(z) = 0$ (cf. § 64) of the present developments,

$$u(z) = z\frac{dP_\nu(z)}{dz} - hP_\nu(z) = 0.$$

We may therefore apply (64) and write

$$\int_0^1 x^{2\nu+1} P_\nu{}^2(zx)\,dx = \frac{1}{2z}\left[\frac{\partial P}{\partial x}\cdot\frac{\partial P}{\partial z} - P\frac{\partial^2 P}{\partial z \partial x}\right]_{x=1}$$

or since

$$\frac{\partial P}{\partial x} = \frac{z}{x}\frac{\partial P}{\partial z}, \qquad \frac{\partial^2 P}{\partial z \partial x} = \frac{z}{x}\frac{\partial^2 P}{\partial z^2},$$

we have

(109)
$$\begin{aligned} \int_0^1 x^{2\nu+1} P_\nu{}^2(zx)\,dx &= \frac{1}{2z}\left[\frac{z}{x}\left(\frac{\partial P}{\partial z}\right)^2 - \frac{1}{x}P\frac{\partial P}{\partial z} - \frac{z}{x}P\frac{\partial^2 P}{\partial z^2}\right]_{x=1} \\ &= \frac{1}{2z}\left[z\left(\frac{\partial P}{\partial z}\right)^2 + 2\nu P\frac{\partial P}{\partial z} + zP^2\right]_{x=1} \end{aligned}$$

by (105).

Thus, if $h = \pm\infty$ so that $u(z)$ becomes simply $P_\nu(z)$, we have

(110)
$$\int_0^1 x^{2\nu+1} P_\nu{}^2(\lambda_n x)\,dx = \tfrac{1}{2}\left(\frac{\partial}{\partial z}P_\nu(z)\right)^2_{z=\lambda_n} = \tfrac{1}{2}u'(\lambda_n)^2$$

and, since we then have by (105),

$$\frac{u''(\lambda_n)}{u'(\lambda_n)} = \frac{P_\nu''(\lambda_n)}{P_\nu'(\lambda_n)} = -\frac{2\nu+1}{\lambda_n}$$

it appears that if we wish to apply (77) in the study of the function $\varphi(n, \alpha, t)$ of the present developments we may take at once $\psi(z) = 1/z^{2\nu+1}$ and $\theta(z)$ such that

(111)
$$\theta'(z) = 2z^{2\nu+1}P(\alpha z)\int_0^t (\alpha + t)^{2\nu+1} P\{(\alpha + t)z\}\,dt; \qquad P = P_\nu.$$

On the other hand, if h be finite so that $u(z) = zP'(z) - hP(z)$, we shall have by (109)

$$\int_0^1 x^{2\nu+1}P^2(\lambda_n x)dx = \frac{P^2(\lambda_n)}{2\lambda_n^2}\{h(2\nu + h) + \lambda_n^2\}. \tag{112}$$

Now, we have by (105)

$$u'(z) = zP''(z) + (1 - h)P'(z) = -(2\nu + h)P'(z) - zP(z),$$

$$u''(z) = -(2\nu + h)P''(z) - zP'(z) - P(z) = \frac{(2\nu + 1)(2\nu + h)}{z}P'(z)$$
$$- zP'(z) + (2\nu + h - 1)P(z).$$

Whence,

$$u'(\lambda_n) = -\frac{P(\lambda_n)}{\lambda_n}\{h(2\nu + h) + \lambda_n^2\},$$

$$u''(\lambda_n) = \frac{P(\lambda_n)}{\lambda_n^2}\{h(2\nu + 1)(2\nu + h) + (2\nu - 1)\lambda_n^2\},$$

so that

$$\frac{u'(\lambda_n)^2}{\int_0^1 x^{2\nu+1}P^2(\lambda_n x)dx} = 2\{h(2\nu + h) + \lambda_n^2\}$$

$$\frac{u''(\lambda_n)}{u'(\lambda_n)} = -\frac{1}{\lambda_n}\frac{h(2\nu + 1)(2\nu + h) + (2\nu - 1)\lambda_n^2}{h(2\nu + h) + \lambda_n^2} = -\frac{2\nu + 1}{\lambda_n} + \frac{2\lambda_n^2}{h(2\nu + h) + \lambda_n^2}.$$

Thus, in order to satisfy the conditions relative to $\psi(z)$ in the present case we should take it so that

$$\frac{\psi'(z)}{\psi(z)} = -\frac{2\nu + 1}{z} + \frac{2z^2}{h(2\nu + h) + z^2}.$$

Let us therefore take

$$\psi(z) = \frac{h(2\nu + h) + z^2}{z^{2\nu+1}}$$

in which case it appears that we may take $\theta(z)$ as before, viz., such that equation (111) is satisfied.

Now we have from (105)

$$\frac{\partial}{\partial z}\left\{z^{2\nu+1}\frac{\partial P(\alpha z)}{\partial z}\right\} + z^{2\nu+1}\alpha^2 P(\alpha z) = 0$$

with a similar equation obtained by replacing α by $\alpha + t$. Hence,

$$[(\alpha + t)^2 - \alpha^2]z^{2\nu+1}P(\alpha z)P\{(\alpha + t)z\}$$
$$= \frac{\partial}{\partial z}\left[z^{2\nu+1}\left(\frac{\partial P(\alpha z)}{\partial z}P\{(\alpha + t)z\} - \frac{\partial P\{(\alpha + t)z\}}{\partial z}P(\alpha z)\right)\right].$$

Placing for convenience $\alpha + t = \beta$ and letting accents represent differentiation with respect to z, we may therefore write

$$\int_0^z z^{2\nu+1}P(\alpha z)P(\beta z)dz = \frac{z^{2\nu+1}}{\beta^2 - \alpha^2}[P'(\alpha z)P(\beta z) - P'(\beta z)P(\alpha z)]$$

so that both when h is finite and when infinite we may take

$$\theta(z) = 2z^{2\nu+1}\int_0^t \frac{\beta^2}{\beta^2 - \alpha^2}[P'(\alpha z)P(\beta z) - P'(\beta z)P(\alpha z)]dt. \tag{113}$$

Upon noting the analytic properties of the functions $\psi(z)$ corresponding to the two above mentioned cases and of the function $\theta(z)$, it appears, upon applying (77) that the integral (36) of the present developments will be given by the expression

$$\frac{1}{\pi i}\int_0^t \frac{\beta^{2\nu+1}}{\beta^2 - \alpha^2}dt \int_{C_n} \frac{P'(\alpha z)P(\beta z) - P'(\beta z)P(\alpha z)}{P^2(z)}dz \tag{114}$$

or

$$\frac{1}{\pi i}\int_0^t \frac{\beta^{2\nu+1}}{\beta^2 - \alpha^2}dt \int_{C_n} \frac{h(2\nu + h) + z^2}{[zP'(z) - hP(z)]^2}[P'(\alpha z)P(\beta z) - P'(\beta z)P(\alpha z)]dz \tag{115}$$

according as $u(z) = P(z)$ or $u(z) = zP'(z) - hP(z)$.

It is to be noted also that in the developments (106) we shall have by (110) and (112)

$$q_n = \frac{2}{P'(\lambda_n)^2}\int_0^1 f(x)x^{2\nu+1}P(\lambda_n x)dx$$

or

$$q_n = \frac{2\lambda_n^2}{[h(2\nu + h) + \lambda_n^2]P^2(\lambda_n)}\int_0^1 f(x)x^{2\nu+1}P(\lambda_n x)dx$$

according as the quantities $\lambda_1, \lambda_2, \cdots$ are the roots of $P(z) = 0$ or

$$zP'(z) - hP(z) = 0.$$

These results premised, let us now consider the rectangle in the z plane whose vertices are the points $z = ij$, $z = k + ij$, $z = k - ij$, $z = -ij$, j being any positive quantity arbitrarily large and k being any positive quantity lying between λ_n and λ_{n+1} where $\lambda_1, \lambda_2, \cdots$ represent the successive positive roots of the equation $P(z) = 0$ or $zP'(z) - hP(z) = 0$ according as we are dealing with (114) or (115). From the boundary of this rectangle let us exclude the point $z = 0$ by means of a small semicircle of radius η and let us now take the resulting contour in part rectangular and in part semicircular as the contour of integration C_n.

Now, the function appearing in the integrand of either (114) or (115) is an odd function of z and hence the two portions of these integrals extended along the y axis mutually destroy each other, while in either case the portion extended

along the semicircle may be made arbitrarily small with η unless in (115) we have $h = 0$. In this exceptional case the integrand of (115) has a pole of the first order at the point $z = 0$ and hence, upon applying Cauchy's integral theorem, the value of the contribution to (115) arising from the semicircle in question becomes

$$-(2\nu + 2)\int_0^t \beta^{2\nu+1} dt = \alpha^{2\nu+2} - \beta^{2\nu+2}.$$

In order to discuss the remaining portions of the integrals (114) and (115) we shall now make use of the following established result:[21]

"Representing by $J_\nu(z)$ Bessel's function of order ν we shall have when $\nu > -\frac{1}{2}$ and z has any value except zero whose real part is positive or zero

$$(116) \qquad J_\nu(z) = -\frac{1}{\sqrt{2\pi z}}[S_1(z)e^{i(z-[(2\nu-1)/4]\pi)} - S_2(z)e^{-i(z-[(2\nu-1)/4]\pi)}],$$

where

$$(117) \qquad \begin{aligned} S_1(z) &= \frac{1}{\Gamma(\nu + \frac{1}{2})}\int_0^\infty e^{-s}s^{\nu-1/2}\left(1 - \frac{s}{2iz}\right)^{\nu-1/2} ds, \\ S_2(z) &= \frac{1}{\Gamma(\nu + \frac{1}{2})}\int_0^\infty e^{-s}s^{\nu-1/2}\left(1 + \frac{s}{2iz}\right)^{\nu-1/2} ds, \end{aligned}$$

and where, when $|z|$ is sufficiently large, these expressions S_1 and S_2 may be expanded into the forms

$$(118) \qquad \begin{aligned} S_1(z) &= \sum_{n=0}^{m-1}\frac{\Gamma(\nu + \frac{1}{2} + n)}{\Gamma(n + 1)\Gamma(\nu + \frac{1}{2} - n)}\frac{1}{(-2iz)^n} + \Theta_1(z, \nu), \\ S_2(z) &= \sum_{n=0}^{m-1}\frac{\Gamma(\nu + \frac{1}{2} + n)}{\Gamma(n + 1)\Gamma(\nu + \frac{1}{2} - n)}\frac{1}{(2iz)^n} + \Theta_2(z, \nu), \end{aligned}$$

in which m is any positive integer and in which the expressions $\Theta_1(z, \nu)$ and $\Theta_2(z, \nu)$ become infinitesimals of order as high as the mth when $|z| = \infty$ and, at least when $\nu > +\frac{1}{2}$, possess first derivatives which as $|z| = \infty$ become infinitesimals of order as high as the $(m + 1)$st."

Placing $-i = e^{-\pi i/2}$ in (116) we obtain

$$J_\nu(z) = \frac{1}{\sqrt{2\pi z}}[S_1(z)e^{i(z-[(2\nu-1)/4]\pi)} + S_2(z)e^{-i(z-[(2\nu-1)/4]\pi)}].$$

Whence, upon expanding and making use of (104), we have

$$(119) \qquad \begin{aligned} P_\nu(z) = \frac{1}{2^{\nu+(1/2)}\sqrt{2\pi}}\Big[&\{S_1(z) + S_2(z)\}\cos\left(z - \frac{2\nu + 1}{4}\pi\right) \\ &+ i\{S_1(z) - S_2(z)\}\sin\left(z - \frac{2\nu + 1}{4}\pi\right)\Big], \end{aligned}$$

[21] Cf. H. WEBER, *Math. Annalen*, Vol. 37 (1890), pp. 404–416. The facts which we shall state regarding the derivatives of $\Theta_1(z, \nu)$ and $\Theta_2(z, \nu)$ are not explicitly obtained by WEBER, but follow at once from his analysis.

so that by (118) we may write when $\nu > \frac{1}{2}$ and when the real part of z is positive (or zero)

$$(120)\quad P_\nu(z) = \sqrt{\frac{2}{\pi}}\frac{1}{z^{\nu+(1/2)}}\left[\{1+\epsilon(z,\nu)\}\cos\left(z-\frac{2\nu+1}{4}\pi\right)\right. \\ \left. + \zeta(z,\nu)\sin\left(z-\frac{2\nu+1}{4}\pi\right)\right],$$

where the functions $\epsilon(z, \nu)$ and $\zeta(z, \nu)$ become infinitesimals of at least the second and first orders respectively as $|z| = \infty$ and possess first derivatives which as $|z| = \infty$ become infinitesimals of at least the third and second orders respectively.

Moreover, by use of the relation $P_\nu(z) = (2\nu + 2)P_{\nu+1}(z) - z^2P_{\nu+2}(z)$, we may readily show that (120) holds true for all values of ν for which $P_\nu(z)$ has a meaning—*i. e.*, unless ν is a negative integer.

Furthermore, since $P'(z) = -zP_{\nu+1}(z)$ we see that unless ν is a negative integer we may write

$$(121)\quad P_\nu'(z) = -\sqrt{\frac{2}{\pi}}\frac{1}{z^{\nu+(1/2)}}\left[\{1+\eta(z,\nu)\}\sin\left(z-\frac{2\nu+1}{4}\pi\right)\right. \\ \left. + \theta(z,\nu)\cos\left(z-\frac{2\nu+1}{4}\pi\right)\right]$$

where $\eta(z, \nu)$ and $\theta(z, \nu)$ have the properties mentioned above for $\epsilon(z, \nu)$ and $\zeta(z, \nu)$ respectively.

Equations (120) and (121) having been obtained, we return now to the discussion of (114) and (115) when the indicated integration is extended over the portions of C_n remaining after removing the semicircle of radius η and the portions of the y axis. Placing for brevity

$$a = \frac{2\nu+1}{4}\pi, \qquad c = \sqrt{\frac{2}{\pi}},$$

we have by (120) and (121) for all values of z upon these portions of C_n, unless $\alpha = 0$ or $\beta = 0$,

$$(122)\quad P(\alpha z) = \frac{c}{(\alpha z)^{\nu+(1/2)}}[\{1+\epsilon(\alpha z)\}\cos(\alpha z - a) + \zeta(\alpha z)\sin(\alpha z - a)],$$

$$(123)\quad P'(\alpha z) = \frac{-c}{z(\alpha z)^{\nu-(1/2)}}[\{1+\eta(\alpha z)\}\sin(\alpha z - a) + \theta(\alpha z)\cos(\alpha z - a)],$$

$$(124)\quad P(\beta z) = \frac{c}{(\beta z)^{\nu+(1/2)}}[\{1+\epsilon(\beta z)\}\cos(\beta z - a) + \zeta(\beta z)\sin(\beta z - a)],$$

$$(125)\quad P'(\beta z) = \frac{-c}{z(\beta z)^{\nu-(1/2)}}[\{1+\eta(\beta z)\}\sin(\beta z - a) + \theta(\beta z)\cos(\beta z - a)],$$

and, excluding the case in which $\alpha = 0$, we observe that in applying Theorem I of § 51 to the integrals (114) and (115) in question the values of α, t and $\beta = \alpha + t$ with which we shall be concerned are such that

$$0 < a' < \alpha < b' < 1,$$

$$-\alpha \leqq t \leqq 1 - \alpha,$$

$$0 < a' < \alpha \leqq \alpha + \beta \leqq 1 + \alpha < 1 + b' < 2,$$

while in applying Theorems VI and VIII of § 55 for the case in which $\alpha = 1$, we shall have $-1 \leqq t \leqq 0$, $1 \leqq \alpha + \beta \leqq 2$.

However, when $t = -\alpha$ we have $\beta = \alpha + t = 0$ so that expressions (114) and (115) cannot be used for all the values of β with which we shall be concerned. Let us therefore exclude for the present the value $t = -\alpha$ from our investigations, treating it later as one of the exceptional values of the type mentioned in remark (1) of § 56. Thus, representing by ξ an arbitrarily small positive quantity, we proceed to study the integrals (114) and (115) for all values of α, t and β satisfying the relations

$$0 < a' < \alpha < b' < 1,$$

$$-\alpha + \xi \leqq t \leqq 1 - \alpha, \tag{126}$$

$$0 < a' < \alpha < \alpha + \xi \leqq \alpha + \beta \leqq 1 + \alpha < 1 + b' < 2,$$

or

$$\alpha = 1,$$

$$-1 + \xi \leqq t \leqq 0, \tag{127}$$

$$1 \leqq \alpha + \beta \leqq 2.$$

From (122), (123), (124) and (125) we find upon performing the indicated multiplications that

$$\frac{\beta^{2\nu+1}}{\beta^2 - \alpha^2}[P'(\alpha z)P(\beta z) - P'(\beta z)P(\alpha z)] = -\frac{c^2}{z^{2\nu+1}(\beta^2 - \alpha^2)}\left(\frac{\beta}{\alpha}\right)^{\nu+(1/2)}$$

$$\times [\{\alpha[1 + \epsilon(\beta z)][1 + \eta(\alpha z)] - \beta\theta(\beta z)\zeta(\alpha z)\} \sin(\alpha z - a) \cos(\beta z - a)$$

$$- \{\beta[1 + \epsilon(\alpha z)][1 + \eta(\beta z)] - \alpha\theta(\alpha z)\zeta(\beta z)\} \sin(\beta z - a) \cos(\alpha z - a)$$

$$+ \{\alpha[1 + \eta(\alpha z)]\zeta(\beta z) - \beta[1 + \eta(\beta z)]\zeta(\alpha z)\} \sin(\alpha z - a) \sin(\beta z - a)$$

$$+ \{\alpha[1 + \epsilon(\beta z)]\theta(\alpha z) - \beta[1 + \epsilon(\alpha z)]\theta(\beta z)\} \cos(\alpha z - a) \cos(\beta z - a)]$$

$$= \frac{-c^2}{z^{2\nu+1}(\beta^2 - \alpha^2)}\left(\frac{\beta}{\alpha}\right)^{\nu+(1/2)} [A \sin(\alpha z - a) \cos(\beta z - a)$$

$$+ B \sin(\beta z - a) \cos(\alpha z - a) + C \sin(\alpha z - a) \sin(\beta z - a)$$

$$+ D \cos(\alpha z - a) \cos(\beta z - a)]$$

$$= \frac{-c^2}{2z^{2\nu+1}(\beta^2-\alpha^2)}\left(\frac{\beta}{\alpha}\right)^{\nu+(1/2)}[(A-B)\sin(\alpha-\beta)z+(A+B)\sin(\alpha+\beta-2a)z$$
$$+(C+D)\cos(\alpha-\beta)z+(D-C)\cos(\alpha+\beta-2a)z],$$

where A, B, C and D are used for brevity to denote the respective coefficients given above of $\sin(\alpha z-a)\cos(\beta z-a)$, etc.

Now, we may write

$$A-B=(\alpha+\beta)[1+p_1(z)], \qquad A+B=(\alpha-\beta)[1+p_2(z)],$$
$$C+D=(\alpha-\beta)p_3(z), \qquad C-D=(\alpha-\beta)p_4(z),$$

where, recalling the properties of the functions $\epsilon(\alpha z)$, $\epsilon(\beta z)$, etc., we see that for all values of α, t and β in (126) and (127) and for all values of z now under consideration the functions $p_1(z)$, $p_2(z)$, $p_3(z)$ and $p_4(z)$ are finite and vanish uniformly like $1/z^2$, $1/z^2$, $1/z$ and $1/z$ respectively as $|z|=\infty$. We note also that these functions may if desired be put in the forms

$$p_1(z)=\frac{b_0}{z^2}, \qquad p_2(z)=\frac{c_0}{z^2}, \qquad p_3(z)=\frac{d_0}{z}+\frac{e_0}{z^2}, \qquad p_4(z)=\frac{f_0}{z}+\frac{g_0}{z^2}$$

in which d_0 and f_0 depend only upon α and β and are finite for all values of α and β in (126) and (127) while b_0, c_0, e_0 and g_0 depend also upon z but for all values of α and β under consideration and for all values of z under consideration and such that $|z|>Q=$ constant, are less in absolute value than certain constants independent of α, β and z. For the same values of z we have also

$$P(z)=\frac{c}{z^{\nu+(1/2)}}[1+\epsilon(z)][\cos(z-a)+\omega(z)\sin(z-a)]$$

$$zP'(z)-hP(z)=\frac{-c}{z^{\nu-(1/2)}}\left[1+\eta(z)+\frac{h}{z}\zeta(z)\right][\sin(z-a)+\overline{\omega}(z)\cos(z-a)],$$

where

$$\omega(z)=\frac{\zeta(z)}{1+\epsilon(z)}, \qquad \overline{\omega}(z)=\frac{\theta(z)+\dfrac{h}{z}[1+\epsilon(z)]}{1+\eta(z)+\dfrac{h}{z}\zeta(z)}.$$

Whence, upon recalling that $\alpha-\beta=-t$, we see that whether we are dealing with (114) or (115), the portions of the integral arising from the part C_n' of C_n now under discussion will be of the form

$$\frac{1}{2\pi i}\int_0^t\left(\frac{\beta}{\alpha}\right)^{\nu+(1/2)}dt\int_{C_n'}\frac{1+q_1(z)}{E^2}\frac{\sin tz}{t}dz$$
$$+\frac{1}{2\pi i}\int_0^t\left(\frac{\beta}{\alpha}\right)^{\nu+(1/2)}\frac{dt}{\alpha+\beta}\int_{C_n'}\frac{\sin[(\alpha+\beta)z-2a]}{E^2}dz$$

$$(128)\qquad + \frac{1}{2\pi i}\int_0^t \left(\frac{\beta}{\alpha}\right)^{\nu+(1/2)} \frac{dt}{\alpha+\beta}\int_{C_n'} \frac{q_2(z)\sin[(\alpha+\beta)z-2a]}{E^2}\,dz$$

$$+ \frac{1}{2\pi i}\int_0^t \left(\frac{\beta}{\alpha}\right)^{\nu+(1/2)} \frac{dt}{\alpha+\beta}\int_{C_n'} \frac{q_3(z)\cos tz}{E^2}\,dz$$

$$+ \frac{1}{2\pi i}\int_0^t \left(\frac{\beta}{\alpha}\right)^{\nu+(1/2)} \frac{dt}{\alpha+\beta}\int_{C_n'} \frac{q_4(z)\cos[(\alpha+\beta)z-2a]}{E^2}\,dz,$$

where the functions $q_1(z)$, $q_2(z)$, $q_3(z)$ and $q_4(z)$ (like the functions $p_1(z)$, $p_2(z)$, etc.) may be put into the forms b_1/z^2, c_1/z^2, $d_1/z + e_1/z^2$ and $f_1/z + g_1/z^2$ respectively and where

$$(129)\qquad \begin{aligned} E &= \cos(z-a) + \omega(z)\sin(z-a), \\ E &= \sin(z-a) + \overline{\omega}(z)\cos(z-a), \end{aligned}$$

according as we are dealing with (114) or (115).

Considering first the portion of C_n' consisting of one of the lines parallel to the x axis, we readily obtain as in § 64 the fact that for all values of α, β and t in (126) each of the integrals in (128) when extended over the line in question approaches uniformly the limit zero as $j = \infty$. Thus we have merely to consider (128) in which $z = k + iy$ and C_n' is understood to extend from $y = -\infty$ to $y = +\infty$ along the line $z = k + iy$.

Now, from the manner in which k is to be chosen, we see from (129) that we may take $k = n\pi + a$ or $k = n\pi/2 + a$; (n = *positive integer*) according as we are dealing with (114) or (115). In either case, equations (129) are such that

$$\frac{1}{E^2} = 1 + \frac{\gamma}{z}\tanh y + \frac{\delta}{z^2},$$

where γ is independent of z while δ depends upon z but has a modulus which for all values of z under consideration is less than a certain quantity M.

Thus, (128) may be written in the form

$$\frac{1}{2\pi}\int_0^t \left(\frac{\beta}{\alpha}\right)^{\nu+(1/2)} dt \int_{-\infty}^{\infty}\left(1+\frac{\gamma}{z}\tanh y + \frac{b_2}{z^2}\right)\frac{\sin tz}{t\cosh^2 y}\,dy$$

$$+ \frac{1}{2\pi}\int_0^t \left(\frac{\beta}{\alpha}\right)^{\nu+(1/2)} \frac{dt}{\alpha+\beta}\int_{-\infty}^{\infty}\left(1+\frac{\gamma}{z}\tanh y + \frac{b_2}{z^2}\right)\frac{\sin[(\alpha+\beta)z-2a]}{\cosh^2 y}\,dy$$

$$(130)\qquad + \frac{1}{2\pi}\int_0^t \left(\frac{\beta}{\alpha}\right)^{\nu+(1/2)} \frac{dt}{\alpha+\beta}\int_{-\infty}^{\infty}\left\{\left(\frac{d_1}{z}+\frac{e_1}{z^2}\right)\cos tz + \left(\frac{f_1}{z}+\frac{g_1}{z^2}\right)\cos[(\alpha+\beta)z-2a]\right\}\frac{dy}{\cosh^2 y},$$

where b_2 has the properties mentioned above of b_1.

Considering first the terms of (130) which have z^2 in their denominator, we have but to refer to the discussion of similar terms in (89) in order to see that for all values of α, β and t in (126) these terms have uniformly the limit zero when $k = \infty$. The same is true also of the term

$$\int_0^t \left(\frac{\beta}{\alpha}\right)^{\nu+(1/2)} \frac{e_1 dt}{\alpha+\beta} \int_{-\infty}^{\infty} \frac{\cos tz}{z \cosh^2 y} dy,$$

since we have $\cos tz = \cos tk \cosh ty - i \sin tk \sinh ty$ and we know that when $|t| < 1$ (as is the case in (126)) the integrals

$$\int_{-\infty}^{\infty} \frac{\cosh ty}{\cosh^2 y} dy, \qquad \int_{-\infty}^{\infty} \frac{|\sinh ty|}{\cosh^2 y} dy$$

have a meaning.

Thus, (130) reduces to

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^t \left(\frac{\beta}{\alpha}\right)^{\nu+(1/2)} dt \int_{-\infty}^{\infty} \left(1+\frac{\gamma}{z}\tanh y\right)\frac{\sin tz}{t\cosh^2 y} dy \\
(131) \qquad &+\frac{1}{2\pi}\int_0^t \left(\frac{\beta}{\alpha}\right)^{\nu+(1/2)} \frac{dt}{\alpha+\beta}\int_{-\infty}^{\infty}\left(1+\frac{\gamma}{z}\tanh y\right)\frac{\sin[(\alpha+\beta)z-2a]}{\cosh^2 y} dy \\
&+\frac{1}{2\pi}\int_0^t \left(\frac{\beta}{\alpha}\right)^{\nu+(1/2)} \frac{e_1'}{\alpha+\beta}\int_{-\infty}^{\infty}\frac{\cos[(\alpha+\beta)z-2a]}{z\cosh^2 y} dy + \Lambda(\alpha, t, k),
\end{aligned}$$

where e_1' depends only upon α, β and t and is finite for all values of these quantities in (126) and where $\Lambda(\alpha, t, k)$ and also $d\Lambda(\alpha, t, k)/dt$ depends upon α, β, t and z but when considered for all values of α, β and t in (126) may be made (uniformly) as small as we please in absolute value by taking k sufficiently large.

Upon placing $z = k + iy$ and recalling the values which k may assume; also placing for convenience $\alpha + \beta = 2 - \tau$ and dropping those integrals which vanish identically since they are relative to odd functions of y, we thus obtain (131) in the form

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^t \left(\frac{\beta}{\alpha}\right)^{\nu+(1/2)} \frac{\sin kt}{t} dt \int_{-\infty}^{\infty}\frac{\cosh ty}{\cosh^2 y} dy \\
&-\frac{\gamma i}{2\pi}\int_0^t \left(\frac{\beta}{\alpha}\right)^{\nu+(1/2)} \frac{\sin kt}{t} \int_{-\infty}^{\infty}\frac{y\tanh y\cosh ty}{(k^2+y^2)\cosh^2 y} dy \\
&+\frac{\gamma i}{2\pi}\int_0^t \left(\frac{\beta}{\alpha}\right)^{\nu+(1/2)} \cos kt dt \int_{-\infty}^{\infty}\frac{k\tanh y\sinh ty}{(k^2+y^2)t\cosh^2 y} dy \\
(132) \qquad &\pm\frac{1}{2\pi}\int_{2(1-a)}^{\tau} \left(\frac{\beta}{\alpha}\right)^{\nu+(1/2)} \frac{\sin k\tau}{\alpha+\beta} d\tau \int_{-\infty}^{\infty}\frac{\cosh(\alpha+\beta)y}{\cosh^2 y} dy \\
&\mp\frac{\gamma i}{2\pi}\int_{2(1-a)}^{\tau} \left(\frac{\beta}{\alpha}\right)^{\nu+(1/2)} \frac{\sin k\tau}{\alpha+\beta} d\tau \int_{-\infty}^{\infty}\frac{y}{k^2+y^2}\frac{\tanh y\cosh(\alpha+\beta)y}{\cosh^2 y} dy
\end{aligned}$$

$$\mp \frac{\gamma i}{2\pi} \int_{2(1-a)}^{\tau} \left(\frac{\beta}{\alpha}\right)^{\nu+(1/2)} \frac{\cos k\tau}{\alpha+\beta} d\tau \int_{-\infty}^{\infty} \frac{k}{k^2+y^2} \frac{\tanh y \sinh (\alpha+\beta)y}{\cosh^2 y} dy$$

$$\mp \frac{1}{2\pi} \int_{2(1-a)}^{\tau} \left(\frac{\beta}{\alpha}\right)^{\nu+(1/2)} e_1' \frac{\cos k\tau}{\alpha+\beta} d\tau \int_{-\infty}^{\infty} \frac{k \cosh (\alpha+\beta)y}{(k^2+y^2)\cosh^2 y} dy$$

$$\mp \frac{1}{2\pi} \int_{2(1-a)}^{\tau} \left(\frac{\beta}{\alpha}\right)^{\nu+(1/2)} e_1' \frac{\sin k\tau}{\alpha+\beta} d\tau \int_{-\infty}^{\infty} \frac{y \sinh (\alpha+\beta)y}{(k^2+y^2)\cosh^2 y} dy + \Lambda(\alpha, t, k),$$

the upper or lower sign being taken according as we are dealing with (114) or (115).

The expression (132) may, moreover, be used to determine the value of the integrals (114) and (115) corresponding to the case $\alpha = 1$. In fact, when $\alpha = 1$ and β and t are confined by (127) we readily see that each term of (132) continues to have a meaning.

From the properties already found of the integrals in (93) it now appears that the second, third, fifth, sixth, seventh and eighth integrals of (132) when considered for all values of α, β and t in (126) or (127) have uniformly the limit zero as $k = \infty$, while if we treat the first integral as we treated the first integral of (93), remembering here that $\lim\limits_{t=0} (\beta/\alpha)^{\nu+(1/2)} = 1$, we find that when $k = \infty$ this integral behaves precisely as the indicated integral of (93)—*i. e.*, approaches the limit $\frac{1}{2}$ or $-\frac{1}{2}$ according as $t > 0$ or $t < -0$.

Similarly, the integral

$$\frac{1}{2\pi} \int_{2(1-a)}^{\tau} \left(\frac{\beta}{\alpha}\right)^{\nu+(1/2)} \frac{\sin k\tau}{\alpha+\beta} d\tau \int_{-\infty}^{\infty} \frac{\cosh (\alpha+\beta)y}{\cosh^2 y} dy,$$

like the fourth integral of (93), has uniformly the limit zero if $a' < \alpha < b'$ while if $\alpha = 1$ it has the limit $\frac{1}{2}$.

Whence, if we are dealing with values of α, β and t satisfying (126), the expression (132) converges uniformly to the limit $\frac{1}{2}$ or $-\frac{1}{2}$ when $k = \infty$ according as $t > 0$ or $t < -0$, while if $\alpha = 1$ and β and t have values consistent with (127) the same expression has the limit 0 or 1 according as we are dealing with (114) or (115).

Thus, exception being made of the case $h = 0$ in the integral (115), the integrals (114) and (115) satisfy relation (I) of Theorem I, § 51 provided, however, that t has only those values for which $-\alpha + \xi \leqq t \leqq 1 - \alpha$; $\xi > 0$. Moreover, when $\alpha = 1$, relation $(\mathrm{I})_b$ of Theorem VI, § 55 is satisfied for the same values of t and in this relation we have in the present instance $G_2 = 0$ or $G_2 = 1$ according as we are dealing with (114) or (115).

Again, if $h = 0$ in (115) the limit approached by this expression as $k = \infty$ $(a' < \alpha < b')$ will be $\frac{1}{2} + (\alpha^{2\nu+2} - \beta^{2\nu+2})$ or $-\frac{1}{2} + (\alpha^{2\nu+2} - \beta^{2\nu+2})$ according as $t > 0$ or $t < 0$, it being understood as before that $-\alpha + \xi \leqq t \leqq 1 - \alpha$.

Likewise, if $\alpha = 1$, other conditions remaining the same, the limit approached

by (115) as $k = \infty$ will be $-1 + (\alpha^{2\nu+2} - \beta^{2\nu+2})$. In both the cases which thus arise when $h = 0$ we evidently meet with an application of the third general remark of § 56 and we shall make this application presently.

Turning to the other relations of Theorems I and VI of §§ 51 and 55, we see that in the present developments the function $\varphi(n, \alpha, t)$ is equal to

$$(133) \qquad \frac{1}{2\pi}\frac{\beta^{2\nu+1}}{\beta^2-\alpha^2}\int_{C_n'}\frac{P'(\alpha z)P(\beta z)-P'(\beta z)P(\alpha z)}{P^2(z)}dz$$

in the case of (114) while for (115) the same function reduces to

$$(134) \qquad \frac{1}{2\pi}\frac{\beta^{2\nu+1}}{\beta^2-\alpha^2}\int_{C_n'}\frac{h(2\nu+h)+z^2}{[zP'(z)-hP(z)]^2}[P'(\alpha z)P(\beta z)-P'(\beta z)P(\alpha z)]dz$$

or

$$(135) \qquad -(2\nu+2)\,\beta^{2\nu+1}+\frac{1}{2\pi}\frac{\beta^{2\nu+1}}{\beta^2-\alpha^2}\int_{C_n'}\frac{P'(\alpha z)P(\beta z)-P(\beta z)P'(\alpha z)}{P'(z)^2}dz,$$

according as $h \neq 0$ or $h = 0$.

Now, for values of α, β and t in (126) we may transform (133), (134) and (135) by use of expressions (122), (123), (124) and (125) and thus we find that, exception being made of the term $-(2\nu+2)\beta^{2\nu+1}$ in (135), these expressions all reduce to the sum of the derivatives with respect to t of the expression (132). From this it follows directly upon using the lemmas of the Appendix that the above expressions satisfy relations II and III of Theorem I, § 51; also that when $\alpha = 1$ conditions $(\mathrm{II})_b$ and $(\mathrm{III})_b$ of Theorem VI of § 55 are satisfied, it being understood throughout as before that we are dealing only with values of t such that $-\alpha + \xi \leqq t \leqq 1 - \alpha$; $\xi > 0$.

Moreover, if we affect each of the terms of (132) by the operation

$$\frac{1}{n}\sum_{n=0}^{n},$$

understanding that absolute values are taken under the various integral signs, it appears as in the study of (93) that when $-\alpha + \xi \leqq t \leqq 1 - \alpha$ $(\xi > 0)$ relation (II)$'$ of § 52 is satisfied, as also $(\mathrm{II})_b'$ $(b = 1)$ of § 55.

It remains, then, merely to consider the integrals (114) and (115) when t takes values such that $-\alpha \leqq t \leqq -\alpha + \xi$ $(\xi > 0)$ and for this it becomes necessary, as already noted, to use some other expressions for $P(\beta z)$ and $P'(\beta z)$ than (124) and (125), since β now takes values indefinitely near to zero.

Considering, then, that $t = -\alpha$ is one of the exceptional points of the type mentioned in remark (1) of § 56 it will now suffice for the application of Theorems I, II, VI and VIII of §§ 51–55 that such additional conditions be placed upon $f(x)$ that when either of the expressions (133), (134) or (135) is multiplied by $f(\alpha + t)$ the absolute value of the product, when considered for values of t such that

$-\alpha \leqq t \leqq -\alpha + \xi$ and for all values of n, may be made uniformly small with ξ, this being true when $a' < \alpha < b'$ and when $\alpha = 1$.

Let us now divide C_n' into two portions C_n'' and C_n''' the first of these comprising that portion of the line $z = k + iy$ for which $|y| < \eta$, where η is an arbitrarily small positive quantity and the second comprising all other portions of C_n'.

As regards the expressions (133), (134) and (135) when the integration is performed over C_n'', we have but to make use of the well known formula

$$P_\nu(z) = \frac{1}{2^\nu \sqrt{\pi}\Gamma(\nu + \frac{1}{2})} \int_0^\pi \sin^{2\nu} \varphi \cos (z \cos \varphi) d\varphi; \qquad \nu > -\tfrac{1}{2}$$

to see that when $|\beta| < \xi$ the same expressions (C_n' now replaced by C_n'') are each of the form $\beta^{2\nu+1}G(\alpha, \beta, n, \xi, \eta)$ where $G(\alpha, \beta, n, \xi, \eta)$ is less in absolute value than a constant independent of α, β and n.

In order to study the same expressions when the integration is performed over C_n''' we first make the following observations:

Let us write (120) in the form

$$(136) \qquad P_\nu(z) = \sqrt{\frac{2}{\pi}} \frac{\cos\left(z - \frac{2\nu + 1}{4}\pi\right)}{z^{\nu+(1/2)}} A(z, \nu),$$

so that

$$A(z, \nu) = \{1 + \epsilon(z, \nu)\} + \zeta(z, \nu) \tan\left(z - \frac{2\nu + 1}{4}\pi\right).$$

For all values of z (real part > 0) lying upon C_n''' and of modulus greater than some fixed value $z_0 > 0$ we see that $A(z, \nu)$ remains less in absolute value than a constant M_1. Moreover, if ν ($\nu \neq$ neg. integer) has any value except one of the form $\frac{1}{2}(1 \pm 4n)$; $n = 0, 1, 2, 3, \cdots$, the same expression when considered for values of z (real or complex) as near to zero as we please remains less in absolute value than a constant M_2 provided $\nu \geqq \frac{1}{2}$. In fact, it appears from (136) that as $z = 0$, $A(z, \nu)$ will tend to zero like $z^{\nu+(1/2)}$, since from (104) we have

$$\lim_{z=0} P_\nu(z) = \frac{1}{2^\nu\Gamma(\nu + 1)}; \qquad \nu > -1.$$

Whence, if ν has any value $\geqq -\frac{1}{2}$ except one of the form $\frac{1}{2}(4n + 1)$; $n = 0, 1, 2, \cdots$, we may write for all values of z upon C_n'''

$$P_\nu(z) = \frac{\cos\left(z - \frac{2\nu + 1}{4}\pi\right)}{z^{\nu+(1/2)}} B(z, \nu),$$

where $B(z, \nu)$ remains less in absolute value than a constant (independent of z).

Similarly, if ν has any value $> -\frac{1}{2}$ except one of the form $\frac{1}{2}(4n-1)$; $n = 0, 1, 2, \cdots$, we may write for all values of z upon C_n'''

$$P_\nu(z) = \frac{\sin\left(z - \dfrac{2\nu+1}{4}\pi\right)}{z^{\nu+(1/2)}} B(z, \nu)$$

where $B(z, \nu)$ has the properties just mentioned.

It follows that for all values of z (real part > 0) upon C_n''' and for all values however small of the positive quantity β we may write, provided $\nu \geqq -\frac{1}{2}$

$$(137) \qquad P_\nu(\beta z) = \frac{\cos\left(\beta z - \dfrac{2\nu+1}{4}\pi\right)}{(\beta z)^{\nu+(1/2)}} B(\beta z, \nu)$$

or

$$(138) \qquad P_\nu(\beta z) = \frac{\sin\left(\beta z - \dfrac{2\nu+1}{4}\pi\right)}{(\beta z)^{\nu+(1/2)}} B(\beta z, \nu),$$

where for the indicated values of z and β the expression $B(\beta z, \nu)$ remains less in absolute value than a constant independent of both β and z and where the first form can be used in all cases except when $\nu = \frac{1}{2}(4n+1)$; $n = 0, 1, 2, \cdots$, while the second form can be used in all cases except when $\nu = \frac{1}{2}(4n-1)$; $n = 0, 1, 2, \cdots$.

By means of the relation

$$P_\nu'(\beta z) = -\beta^2 z P_{\nu+1}(\beta z)$$

we now obtain, as formulae corresponding to (137) and (138),

$$(139) \qquad \begin{aligned} P_\nu'(\beta z) &= \frac{\sin\left(\beta z - \dfrac{2\nu+1}{4}\pi\right)}{\beta^{\nu-(1/2)} z^{\nu+(1/2)}} B(\beta z, \nu), \\ P_\nu'(\beta z) &= \frac{\cos\left(\beta z - \dfrac{2\nu+1}{4}\pi\right)}{\beta^{\nu-(1/2)} z^{\nu+(1/2)}} B(\beta z, \nu), \end{aligned}$$

where $B(\beta z, \nu)$ has the properties given in connection with (137) and (138) and where the first or else the second formula (and in general both) can be used for any given value of $\nu \geqq -\frac{1}{2}$.

Now, if we use in (133), (134) and (135) the forms (137), (138) and (139) (thus confining ourselves to an integration over C_n''') we find as before that by taking $j = \infty$ the complex integrals become simply those arising when, instead of C_n''', we take as path of integration the line $z = k + iy$, it being understood that the integration now consists of that from $y = -\infty$ to $y = -\eta$

together with that from $y = \eta$ to $y = \infty$. This statement, as in the former case, is seen to be true either when $a' < \alpha < b'$ or $\alpha = 1$. The resulting complex integrals thus take a form analogous to (132) involving real integrals of the form

$$\frac{1}{\beta^{\nu+(1/2)}}\left(\int_{-\infty}^{-\eta} + \int_{\eta}^{\infty}\right)\varphi_1(y)dy, \qquad \frac{1}{\beta^{\nu+(1/2)}}\left(\int_{-\infty}^{-\eta} + \int_{\eta}^{\infty}\right)\varphi_2(y)dy,$$

$$\frac{1}{\beta^{\nu+(1/2)}}\left(\int_{-\infty}^{-\eta} + \int_{\eta}^{\infty}\right)\varphi_3(y)dy,$$

where the expressions $\varphi_1(y)$, $\varphi_2(y)$ and $\varphi_3(y)$ are functions of k, α, β and y in each of which the numerator contains, besides factors whose modulus is always less than a constant, terms in each of which appears one of the factors $\sinh \alpha y$, $\cosh \alpha y$ while the denominator contains $\cosh^2 y$. Thus the integrals in question (aside from the factor $\beta^{-\nu-(1/2)}$ appearing on the outside) are always less in absolute value than a constant independent of α, β and k, it being understood throughout that $a' < \alpha < b'$ or $\alpha = 1$ and $|\beta| < \xi$.

Thus the expressions (104), (105) and (106) when considered for values of β such that $|\beta| < \xi$ are of the form $\beta^{\nu+(1/2)} H(\alpha, \beta, n)$ where $H(\alpha, \beta, n)$ is less in absolute value than a constant independent of α, β and n.

It follows therefore (considering the forms which we have now obtained for the expressions (133), (134) and (135) when the indicated integration is performed either over C_n'' or C_n''') that we shall be able to apply Theorems I, II, VI and VIII of §§ 51–55 to the present developments if we demand (in addition to the conditions placed upon $f(x)$ in the same theorems) that the function $f(\beta)\beta^{\nu+(1/2)}$ be integrable in the neighborhood at the right of the point $\beta = 0$, it being understood also that $\nu > -\frac{1}{2}$. In other words, we need merely make the additional demand that $x^{\nu+(1/2)}f(x)$ be integrable in the neighborhood at the right of the point $x = 0$.

Upon applying Theorems I, II, VI and VII of §§ 51–55 and remarks (1) and (3) of § 56 we thus arrive in summary at the following result:

"If $f(x)$ remains finite throughout the interval (0, 1) with the possible exception of a finite number of points and is such that the integrals

(140) $\displaystyle\int_0^{\epsilon} x^{\nu+(1/2)}|f(x)|dx, \qquad \int_{\epsilon}^{1} |f(x)|dx,$ *ϵ arbitrarily small and positive,*

exist and if $P_\nu(z)$ be the function defined for all values of z and for $\nu > -1$ by the equation (104), then each of the three series:

$$\sum_{n=1}^{\infty} q_n P_\nu(\lambda_n x),$$

$$(2\nu + 2)\int_{\epsilon}^{1} x^{2\nu+1} f(x)dx + \sum_{n=1}^{\infty} q_n' P_\nu(\lambda_n' x),$$

$$\sum_{n=1}^{\infty} q_n'' P_\nu(\lambda_n'' x),$$

in which λ_n, λ_n' and λ_n'' represent respectively the nth positive roots of the equations

$$P_\nu(z) = 0, \qquad P_\nu'(z) = 0, \qquad zP_\nu'(z) - hP_\nu(z) = 0; \qquad h = constant \neq 0$$

and in which

$$q_n = \frac{2}{P_\nu'(\lambda_n)^2} \int_0^1 x^{2\nu+1} f(x) P_\nu(\lambda_n x) dx,$$

$$q_n' = \frac{2}{P_\nu(\lambda_n')^2} \int_0^1 x^{2\nu+1} f(x) P_\nu(\lambda_n' x) dx,$$

$$q_n'' = \frac{2\lambda_n''^2}{\{h(2\nu + h) + \lambda_n''^2\} P_\nu(\lambda_n'')^2} \int_0^1 x^{2\nu+1} f(x) P_\nu(\lambda_n'' x) dx$$

will converge provided $\nu \geqq -\frac{1}{2}$ at any point x $(0 < x < 1)$ in the arbitrarily small neighborhood of which $f(x)$ has limited total fluctuation, and the sum will be

$$\tfrac{1}{2}[f(x - 0) + f(x + 0)].$$

Moreover, the convergence will be uniform to the limit $f(x)$ throughout any interval (a', b') enclosed within a second interval (a_1, b_1) such that $0 < a_1 < a' < b' < b_1 < 1$ provided $f(x)$ is continuous throughout (a', b') inclusive of the end points $x = a'$, $x = b'$ and has limited total fluctuation throughout (a_1, b_1).

Also, if $f(x)$ remains finite throughout the interval $(0, 1)$ with the possible exception of a finite number of points and is such that the integrals (140) exist, then each of the three series above $(\nu \geqq -\frac{1}{2})$ will be summable $(r = 1)$ at any point x $(0 < x < 1)$ at which the limits $f(x - 0)$, $f(x + 0)$ exist and the sum will be

$$\tfrac{1}{2}[f(x - 0) + f(x + 0)].$$

Moreover, the summability will be uniform to the limit $f(x)$ throughout any interval (a', b') such that $0 < a' < b' < 1$ provided that at all points within (a', b') inclusive of the end points $x = a'$, $x = b'$ the function $f(x)$ is continuous.

Under the same conditions for $f(x)$ when considered throughout the *whole* interval $(0, 1)$ the three series $(\nu \geqq -\frac{1}{2})$, when considered for the value $x = 1$, will converge to the respective limits 0, $f(1 - 0)$ and $f(1 - 0)$ provided that $f(x)$ is of limited total fluctuation in the neighborhood at the left of the point $x = 1$.

The same series when considered for the value $x = 1$ will be summable to the respective limits 0, $f(1 - 0)$ and $f(1 - 0)$ whenever $f(1 - 0)$ exists."

67. If we now introduce Bessel functions into this result through the relation

$P_\nu(z) = z^{-\nu}J_\nu(z)$ and then apply the theorem to the function $x^{-\nu}f(x)$ instead of $f(x)$ we obtain the following:

THEOREM. *If $f(x)$ remains finite throughout the interval* $(0, 1)$ *with the possible exception of a finite number of points and is such that the integrals*

$$(141)\qquad \int_0^{\epsilon} x^{\frac{1}{2}}|f(x)|\,dx, \qquad \int_\epsilon^1 |f(x)|\,dx; \qquad \epsilon = \text{arbitrarily small positive constant}$$

exist and if $J_\nu(z)$ be Bessel's function of the first kind of order ν then each of the three series

$$\sum_{n=1}^{\infty} q_n J_\nu(\lambda_n x),$$

$$(2\nu + 2)\int_0^1 x^{2\nu+1}f(x)dx + \sum_{n=1}^{\infty} q_n' J_\nu(\lambda_n' x),$$

$$\sum_{n=1}^{\infty} q_n'' J_\nu(\lambda_n'' x),$$

in which λ_n, λ_n' and λ_n'' represent respectively the nth positive roots of the equations

$$J_\nu(z) = 0,$$

$$\frac{d}{dz}(z^{-\nu}J_\nu(z)) = zJ_\nu'(z) - \nu J_\nu(z) = 0,$$

$$zJ_\nu'(z) - (h + \nu)J_\nu(z) = 0, \qquad h = \text{constant} \neq 0,$$

and in which

$$q_n = \frac{2}{J_\nu'(\lambda_n)^2}\int_0^1 xf(x)J_\nu(\lambda_n x)dx,$$

$$q_n' = \frac{2}{J_\nu(\lambda_n')^2}\int_0^1 xf(x)J_\nu(\lambda_n' x)dx,$$

$$q_n'' = \frac{2\lambda_n''^2}{\{h(2\nu + h) + \lambda_n''^2\}J_\nu(\lambda_n'')^2}\int_0^1 xf(x)J_\nu(\lambda_n'' x)dx$$

will converge provided $\nu > -\frac{1}{2}$ at any point x $(0 < x < 1)$ *in the arbitrarily small neighborhood of which $f(x)$ has limited total fluctuation, and the sum will be*

$$\tfrac{1}{2}[f(x - 0) + f(x + 0)].$$

Moreover, the convergence will be uniform to the limit $f(x)$ throughout any interval (a', b') *enclosed within a second interval* (a_1, b_1) *such that* $0 < a_1 < a' < b' < b_1 < 1$ *provided $f(x)$ is continuous throughout* (a', b') *inclusive of the end points $x = a'$, $x = b'$ and has limited total fluctuation throughout* (a_1, b_1).

Also, if $f(x)$ remains finite throughout the interval $(0, 1)$ *with the possible exception of a finite number of points and is such that the integrals* (141) *exist, then*

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each of the three series above ($\nu > -\frac{1}{2}$) *will be summable* ($r = 1$) *at any point* x ($0 < x < 1$) *at which the limits* $f(x-0)$, $f(x+0)$ *exist and the sum will be*

$$\tfrac{1}{2}[f(x-0) + f(x+0)].$$

Moreover, the summability will be uniform (§ 45) *to the limit* $f(x)$ *throughout any interval* (a', b') *such that* $0 < a' < b' < 1$ *provided that at all points within* (a', b') *inclusive of the end points* $x = a'$, $x = b'$ *the function* $f(x)$ *is continuous.*

Under the same conditions for $f(x)$ *when considered throughout the whole interval* $(0, 1)$, *the three series, when considered for the value* $x = 1$ *will converge to the respective limits* 0, $f(1-0)$ *and* $f(1-0)$ *provided that* $f(x)$ *is of limited total fluctuation in the neighborhood at the left of the point* $x = 1$.

The same series when considered for the value $x = 1$ *will be summable* ($r = 1$) *to the respective limits* 0, $f(1-0)$ *and* $f(1-0)$ *whenever* $f(1-0)$ *exists, it being always assumed that the integrals* (141) *exist.*[22]

3. *The Developments in Terms of Legendre Functions.*

68. We proceed to consider the well known development

$$f(x) = \sum_{n=0}^{\infty} q_n X_n(x); \qquad q_n = \frac{2n+1}{2}\int_{-1}^{1} f(x) X_n(x)\,dx, \tag{142}$$

in which $X_n(x)$ represents the polynomial of Legendre (Zonal Harmonic) of order n. In the notation of § 60 we here have a development of the form (54) in which $H(z, x) = X_z(x)$, $a = -1$, $b = 1$ and in which equation (53) becomes

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial X}{\partial x}\right\} + z(z+1)X = 0.$$

Moreover, since z is to take only integral values, the equation $u(z) = 0$ must

[22] It may be noted that our results, in so far as they concern convergence at a *special* point x ($0 < x < 1$), are not in entire accord with those of DINI ("Serie di Fourier," pp. 266–269). In fact, instead of the existence of the first of the integrals (141) DINI requires that $|f(x)|x^{\nu+\frac{1}{2}-p}$, where p is the greater of the two numbers ν, $\frac{1}{2}$, shall be integrable in the neighborhood at the right of the point $x = 0$. This discrepancy is due chiefly to a slight error occurring in formula (95), p. 237 of DINI's work, the last term of which should contain under the integral sign $e^{-\tau}\tau^{2\nu-1-n}$ instead of $e^{-\tau}\tau^{\nu-\frac{1}{2}-n}$, as appears from the analysis on p. 237. If this formula (95) be altered as just indicated and resulting changes be made on pp. 242, 243, 265–269, we are led to the above theorem. This same theorem, so far as it concerns convergence, is in accord with the results published in recent years by HOBSON (*Proc. London Math. Soc.*, Vol. 7 (1908), pp. 359–388), while, as regards summability, the theorem is in accord with the results of C. N. MOORE (*Trans. Am. Math. Soc.*, Vol. 10 (1909), p. 428).

It may also be remarked at this point that, except in the study of uniform convergence, the results published of late years by HOBSON and others respecting the convergence of Fourier series and other developments in terms of special normal functions were originally obtained rigorously for the first time by DINI—a fact apparently not well understood. See, however, NIELSON, "Handbuch der Theorie der Cylinderfunktionen," p. 353.

here be regarded as given in advance and may be taken for example as

$$u(z) = \sin \pi z = 0.$$

Furthermore, we have in the present instance $K(x) = 1 - x^2$ so that equations (60) become satisfied identically by taking $h' = 0$, $h = 0$. However, since $K(\pm 1) = 0$ it follows that the general formulæ of § 61 for the determination of the integral (36) corresponding to the present development cannot be used. It becomes necessary, therefore, in order to ascertain whether this integral satisfies the conditions of the fundamental Theorem I, § 51, to proceed independently of such formulæ.

Now, the integral (36) here becomes

$$\int_0^t \varphi(n, \alpha, t)dt = \tfrac{1}{2}\sum_{n=0}^{n} (2n+1)X_n(\alpha) \int_0^t X_n(\alpha + t)dt, \tag{143}$$

and hence also

$$\varphi(n, \alpha, t) = \tfrac{1}{2}\sum_{n=0}^{n} (2n+1)X_n(\alpha)X_n(\alpha + t). \tag{144}$$

We proceed to show that the three relations of the general Theorem I of § 51 are satisfied in the present instance, it being understood that we here have $a = -1$, $b = 1$.

The values of α and t with which we are concerned are such that

$$-1 < a' \leqq \alpha \leqq b' < 1,$$

$$-1 \leqq \alpha + t \leqq 1.$$

We may therefore place $\alpha = \cos\theta'$, $\alpha + t = \cos\theta$ in which case we have, as is well known,[23]

$$X_n(\cos\theta)X_n(\cos\theta') = \frac{1}{2\pi}\int_0^{2\pi} X_n(\cos\gamma)d\varphi, \tag{145}$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi')$, it being understood that (θ, φ) and (θ', φ') thus represent the polar spherical coördinates of two points M, M' on the unit sphere, while γ represents the spherical distance between the same points.

Thus we may write

$$\int_0^t \varphi(n, \alpha, t)dt = -\frac{1}{4\pi}\sum_{n=0}^{n} (2n+1)\int_{\theta'}^{\theta} \sin\theta d\theta \int_0^{2\pi} X_n(\cos\gamma)d\varphi$$

or, since

$$\sum_{n=0}^{n} (2n+1)X_n(\cos\gamma) = -\frac{1}{\sin\gamma}\left\{\frac{dX_n}{d\gamma} + \frac{dX_{n+1}}{d\gamma}\right\}, \tag{146}$$

[23] Cf. for example, Todhunter's "Treatise on Laplace's Functions, Lamé's Functions, etc." (London, 1875), § 170.

we have

(147) $$\int_0^t \varphi(n, \alpha, t)dt = \frac{1}{4\pi}\int_{\theta'}^{\theta} \sin\theta d\theta \int_0^{2\pi}\left\{\frac{dX_n}{d\gamma} + \frac{dX_{n+1}}{d\gamma}\right\}\frac{d\varphi}{\sin\gamma}.$$

Whence, if we denote by $d\sigma$ the element of spherical surface and observe that $d\theta$ is negative when t is positive (*i. e.*, when $\theta < \theta'$), while $d\theta$ is positive when t is negative, we may write

(148) $$\int_0^t \varphi(n, \alpha, t)dt = \mp\frac{1}{4\pi}\int\int\left\{\frac{dX_n}{d\gamma} + \frac{dX_{n+1}}{d\gamma}\right\}\frac{d\sigma}{\sin\gamma},$$

in which the upper or lower sign is to be taken according as t is positive or negative and where it is understood that the integration is extended over the zone lying between the parallels $\theta = \theta'$ and $\theta = \theta$.

Let us now choose a new coördinate system (γ, ψ) such that the fixed point $M' \equiv (\theta', \varphi')$ becomes the point $\gamma = 0$, while the great circle through M' tangent to the circle $\theta = \theta'$ determines the points for which $\psi = 0$. Then $d\sigma = \sin\gamma d\gamma d\psi$ so that if we represent by $\gamma_1(\theta')$ the value of γ pertaining to the point upon the circle $\theta = \theta'$ having the (variable) coördinate ψ and agree to place for convenience $Y_n(\cos\gamma) = X_n(\cos\gamma) + X_{n+1}(\cos\gamma)$, we may put the equation (148) into the form

$$\int_0^t \varphi(n, \alpha, t)dt = \mp\frac{1}{4\pi}\int_g^h [Y_n(\cos\gamma)]_{\gamma=0}^{\gamma_1(\theta')}d\psi \pm I_n(t),\text{[24]}$$

in which $g = 0$, $h = \pi$ or $g = \pi$, $h = 2\pi$ according as $\alpha \geqq 0$ or $\alpha < 0$ ($\theta' \leqq \pi/2$ or $\theta' > \pi/2$) and in which $I_n(t)$ is defined in one of two ways as follows:

(*a*) If $\theta < \theta'$ or $\theta \geqq \pi - \theta'$,

(149) $$I_n(t) = \frac{1}{4\pi}\int_c^{\pi - c} [Y_n(\cos\gamma)]_{\gamma=\gamma_2(\theta)}^{\gamma_3(\theta)}d\psi,$$

in which c and $\pi - c$ represent the two values of ψ determined by the planes of the two great circles through the point $\gamma = 0$ tangent to the circle $\theta = \theta$ and in which $\gamma_2(\theta)$ and $\gamma_3(\theta)$ represent the two values of γ pertaining to the points upon the circle $\theta = \theta$ having the common coördinate ψ.

(*b*) If $\theta' < \theta < \pi - \theta'$,

(150) $$I_n(t) = \frac{1}{4\pi}\int_0^{2\pi} Y_n(\cos\gamma_3(\theta))d\psi,$$

in which $\gamma_3(\theta)$ represents the value of γ pertaining to the point upon the circle $\theta = \theta$ having the coördinate ψ.

Upon writing

$$[Y_n(\cos\gamma)]_{\gamma=0}^{\gamma_1(\theta')} + [Y_n(\cos\gamma)]_{\gamma=0}^{\pi} - [Y_n(\cos\gamma)]_{\gamma=\gamma_1(\theta')}^{\pi}$$

[24] We here employ the common notation $[f(x)]_{x=a}^{b} = f(b) - f(a)$.

and observing that $Y_n(\cos 0) = 2$, $Y_n(\cos \pi) = 0$, we thus obtain

$$(151)\qquad \int_0^t \varphi(n, \alpha, t)dt = \pm \tfrac{1}{2} \mp \frac{1}{4\pi}\int_g^h Y_n(\cos \gamma_1(\theta'))d\psi \pm I_n(t).$$

In order to show that relation (I) of § 51 is satisfied it therefore suffices to show that for all values of α and t such that

$$(152)\qquad \begin{cases} -1 < a' \leqq \alpha \leqq b' < 1 \\ -1 - \alpha \leqq t \leqq -\epsilon \quad \text{or} \quad \epsilon \leqq t \leqq 1 - \alpha \qquad (\epsilon > 0) \end{cases}$$

the last two terms of (151) converge ($n = \infty$) uniformly to zero. In doing this we shall make use of the following two fundamental results respecting $Y_n(\cos \gamma)$:

(A) For values of γ in any interval such that $0 < \xi \leqq \gamma \leqq \pi - \xi < \pi$ the expression $Y_n(\cos \gamma)$ converges ($n = \infty$) uniformly to zero.

(B) For all values of n we have uniformly $\lim\limits_{\gamma=\pi-0} Y_n(\cos \gamma) = 0$—*i. e.*, corresponding to an arbitrarily small positive quantity σ, one may determine a second positive quantity ζ *independent of* n and such that $|Y_n(\cos \gamma)| < \sigma$ when

$$\pi - \zeta \leqq \gamma \leqq \pi.$$

The proof of (A) follows directly from the well-known fact[25] that $X_n(\cos \gamma)$ satisfies the indicated relation, while the proof of (B) may be supplied as follows:

From the formula[26]

$$X_n(x) = \frac{1}{\pi}\int_0^\pi [x + \sqrt{x^2 - 1}\cos \varphi]^n d\varphi; \qquad -1 \leqq x \leqq 1$$

we have

$$Y_n(x) = \frac{1}{\pi}\int_0^\pi [1 + x + \sqrt{x^2 - 1}\cos \varphi][x + \sqrt{x^2 - 1}\cos \varphi]^n d\varphi.$$

Whence,

$$|Y_n(x)| \leqq \frac{1}{\pi}\int_0^\pi |1 + x + \sqrt{x^2 - 1}\cos \varphi| d\varphi \leqq |1 + x| + \sqrt{1 - x^2},$$

so that for all values of n we have uniformly

$$\lim_{x=-1+0} Y_n(x) = 0 \qquad \text{or} \qquad \lim_{\gamma=\pi-0} Y_n(\cos \gamma) = 0.$$

Results (A) and (B) being premised, we now turn to the second term appearing on the right in (151) (which term, except for the sign $\mp$, is independent of t, but depends upon α). Let us first confine ourselves to values of α which are positive ($0 < \alpha < b'$). For such a value of α the term in question has the form

$$\mp \frac{1}{4\pi}\int_0^\pi Y_n(\cos \gamma_1)d\psi; \qquad \gamma_1 = \gamma_1(\theta').$$

[25] Cf. for example, Fejér, *Math. Annalen*, Vol. 67 (1909), p. 103.

[26] Cf. for example, Byerly's "Fourier Series, etc.," p. 166.

Omitting the factor $\mp 1/4\pi$, let us write this in the form

$$(153)\quad \int_0^{\eta} Y_n(\cos\gamma_1)d\psi + \int_{\eta}^{\pi/2-\eta} Y_n(\cos\gamma_1)d\psi + \int_{\pi/2-\eta}^{\pi/2+\eta} Y_n(\cos\gamma_1)d\psi + \int_{\pi/2+\eta}^{\pi-\eta} Y_n(\cos\gamma_1)d\psi + \int_{\pi-\eta}^{\pi} Y_n(\cos\gamma_1)d\psi,$$

where η is an arbitrarily chosen, small, positive quantity.

Since $|Y_n(\cos\gamma_1)| \leqq 2$ whatever the values of n and γ_1, the first, third and last terms here appearing may be made arbitrarily small in absolute value with a proper choice of η, this being true not only for special values of n and α, but uniformly for all values of α such as we are considering and for all values of n. After η has once been chosen, the values of γ_1 which enter into the second and fourth terms under the sign of integration are seen (upon reference to the unit sphere) to always be such that $0 < \eta_1 \leqq \gamma_1 \leqq \pi - \eta_1 < \pi$ where η_1 depends only upon η. From result (A) it follows that the same terms approach uniformly $(0 < \alpha < b')$ the limit zero as $n = \infty$.

Thus the second term of (151) comes to have the properties desired.

We proceed to examine the properties of the last term in (151)—*i. e.*, of the expression $I_n(t)$. Since t is confined by relations (152), the angle θ never approaches (as t varies) nearer to θ' (regarded as fixed with α) than some positive quantity κ which, if taken small enough, will be independent of both α and t. With κ thus chosen, it now suffices to show that for all circles θ such that either $0 < \theta < \theta' - \kappa$ or $\theta' + \kappa < \theta < \pi$ the expression $I_n(t)$ converges $(n = \infty)$ uniformly to zero. In showing this we shall find it convenient to divide these circles into three classes as follows:

$$(a)\quad 0 < \theta < \theta' - \kappa,$$

$$(b)\quad \theta' + \kappa < \theta < \pi - \theta',$$

$$(c)\quad \pi - \theta' < \theta < \pi.$$

Also, we shall assume for the present (as above) that $\theta' < \pi/2$ $(\alpha > 0)$.

First, for the circles (a) we have $I_n(t)$ defined by (149) in which $\gamma_2(\theta)$ and $\gamma_3(\theta)$ are such that $\kappa \leqq \gamma_2 \leqq \pi/2$, $\kappa \leqq \gamma_3 \leqq 2\theta' - \kappa < \pi$, while c lies between fixed limits dependent only upon ϵ (as again appears after noting the significance of the various letters upon the unit sphere). Whence, by result (A) we reach the desired result for the circles (a).

As regards the circles (b), let us divide these into two sub-classes as follows:

$$(b)'\quad \pi - \theta' - \eta < \theta < \pi - \theta',$$

$$(b)''\quad \theta' + \kappa < \theta < \pi - \theta' - \eta,$$

where η represents an arbitrarily small positive quantity.

For the circles $(b)'$ we have $I_n(t)$ defined by (150), which may be written in the form

(154) $$\frac{1}{4\pi}\int_{-\omega}^{\pi+\omega} Y_n(\cos\gamma_3)d\psi + \frac{1}{4\pi}\int_{\pi+\omega}^{2\pi-\omega} Y_n(\cos\gamma_3)d\psi,$$

where ω represents an arbitrarily small positive quantity. Now, by choosing η and ω each sufficiently small, all the values of γ_3 entering into the first term of (154) may be brought as near as we please to π, so that in view of result (B), we conclude that the first term of (154) may be made arbitrarily small in absolute value by taking η and ω sufficiently small and that this is true uniformly for all values of n. With η and ω once fixed, we now observe that the values of γ_3 entering into the second term of (154), when considered for the circles $(b)'$, never approach nearer to π than a fixed value independent of θ, while the same values of γ_3 remain different by as much as κ from 0. Hence, for reasons already stated in connection with the circles (a), the second term of (154), when considered for the circles $(b)'$ approaches uniformly the limit 0.

With η fixed as above, let us now consider the circles $(b)''$. Here again we have the form (150) for $I_n(t)$, but the values of γ_3 never approach nearer to π than a certain positive value independent of θ, nor nearer to zero than κ, so that as before we see that uniform convergence is present. In summary, the expression $I_n(t)$ has the desired properties for all the circles (b).

We turn lastly to the circles (c). Let us divide these into two sub-classes as follows:

$$(c)' \quad \pi - \theta' < \theta \leqq \pi - \theta' + \mu,$$

$$(c)'' \quad \pi - \theta' + \mu \leqq \theta < \pi,$$

where μ represents an arbitrarily small positive constant. For the circles $(c)'$ we have $I_n(t)$ defined by (149) and by taking μ sufficiently small the values of γ_3 pertaining to these circles $(c)'$ may be made to differ by as little as we please from π. At the same time, however small μ be taken, we have $\gamma_2 \geqq \kappa > 0$. Whence, using result (B), we see as before that if ν be any preassigned arbitrarily small positive quantity, we may take μ so small that for all the circles $(c)'$ we shall have uniformly $|I_n(t)| < \nu$. With μ thus chosen, let us consider the resulting circles $(c)''$. Here again we are to use the form (149), but the values of γ_2 and γ_3 which enter lie between assignable limits m, n such that $m > 0$, $n < \pi$ $(m = \kappa,\ n = \pi - \mu)$. Hence, for the circles $(c)''$ the expression $I_n(t)$ has the desired properties, and in summary we may say that the same is true for all the circles (c).

Thus, relation (I) of § 51 becomes satisfied for all values of α within the interval $0 \leqq \alpha \leqq b' < 1$. That it is satisfied also when $-1 < a' \leqq \alpha \leqq 0$ may now be inferred as follows:

In the present development we have $\varphi(n, \alpha, t) = \varphi(n, -\alpha, -t)$ and hence

$$(155) \qquad \int_0^t \varphi(n, \alpha, t)dt = \int_0^t \varphi(n, -\alpha, -t)dt = -\int_0^{-t} \varphi(n, -\alpha, t)dt.$$

If α be such that $a' \leqq \alpha \leqq 0$ it follows from what we have already shown that the last member of (155) will converge to the limit $\frac{1}{2}$ or $-\frac{1}{2}$ according as $-1+\alpha \leqq -t \leqq -\epsilon$ or $\epsilon \leqq -t \leqq 1+\alpha$, and that for all such values of α and t the convergence will be uniform. This is, however, the same as saying that for $a' \leqq \alpha \leqq 0$ and $-1-\alpha \leqq t \leqq -\epsilon$ or $\epsilon \leqq t \leqq 1-\alpha$ the first member of (155) shall have the properties desired.

That relation (II) of § 51 is also satisfied in the present developments follows from (151) together with (150). Thus, for all values of α in (152) and for all values of t such that $-\epsilon \leqq t \leqq \epsilon$ we have

$$\left| \int_0^t \varphi(n, \alpha, t)dt \right| \leqq \tfrac{1}{2} + \frac{1}{4\pi}\int_0^{\pi} 2d\psi + \frac{1}{4\pi}\int_0^{2\pi} 2d\psi = 2.$$

With regard to relation (III) of § 51, we note that the function $\varphi(n, \alpha, t)$ of the present development is given by (144). Now, availing ourselves of the formula

$$\frac{1}{2}\sum_{n=0}^{n}(2n+1)X_n(x')X_n(x) = \frac{n+1}{2}\,\frac{X_{n+1}(x')X_n(x) - X_n(x')X_{n+1}(x)}{x'-x}$$

and of the fact that for large values of n the function $X_n(\cos\theta)$ is of the form[27]

$$\left(\frac{2}{n\pi\sin\theta}\right)^{\frac{1}{2}} A(n, \theta); \qquad |A(n, \theta)| < 1$$

provided θ lies in any interval of the form $0 < \xi \leqq \theta \leqq \pi - \xi;\ \xi > 0$, it appears that (III) is here satisfied for all values of α and t such that $-1 < a' \leqq \alpha \leqq b' < 1$ and $-1-\alpha+\xi \leqq t \leqq -\epsilon,\ \epsilon \leqq t \leqq 1-\alpha-\xi\ (\xi > 0)$. Whether the same is also true (as desired by (III)) when t lies in the intervals $-1-\alpha \leqq t \leqq -1-\alpha+\xi$ or $1-\alpha-\xi \leqq t \leqq 1-\alpha$ remains in doubt, thus leading eventually to an application of remark (1) of § 56. Due account of this exceptional character will be taken before the final summary of our results into a theorem.

We turn to the consideration of (143) when $\alpha = \pm 1$. First, if $\alpha = 1$ we have

$$(156) \qquad \begin{aligned} \int_0^t \varphi(n, 1, t)dt &= \tfrac{1}{2}\sum_{n=0}^{n}(2n+1)\int_0^t X_n(1+t)dt \\ &= -\tfrac{1}{2}\sum_{n=0}^{n}\int_0^{\theta} X_n(\cos\theta)\sin\theta d\theta \end{aligned}$$

[27] Cf. FEJÉR, *l. c.*, p. 103.

and we shall now show that for values of t such that $-2+\epsilon \leqq t \leqq -\epsilon$; *i. e.*, of θ such that $0 < \eta \leqq \theta \leqq \pi - \eta$ (η arbitrarily small but > 0) the last member of (156) converges (uniformly) to the value -1 when $n = \infty$, thus satisfying relation $(\mathrm{I})_b$ of § 55 ($G_2 = 1$) when exception is there made of the value $t = -2$ ($\theta = \pi$).

In fact, when $\alpha = 1$ we have $\theta' = 0$, so that in using (147) we have $\gamma = \theta$ while φ becomes independent of θ. Thus we may write

$$(157) \quad \begin{aligned} \int_0^{t'} \varphi(n, 1, t)dt &= \tfrac{1}{2}\int_0^{\theta} \frac{d}{d\theta}[X_n(\cos\theta) + X_{n+1}(\cos\theta)]d\theta \\ &= \tfrac{1}{2}[X_n(\cos\theta) + X_{n+1}(\cos\theta)]_0^{\theta} = -1 + \tfrac{1}{2}Y_n(\cos\theta). \end{aligned}$$

The indicated statement thus follows upon noting the properties already mentioned of $Y_n(\cos\theta)$ when $0 < \eta \leqq \theta \leqq \pi - \eta$ ($\eta > 0$).

Again, if $\alpha = -1$ we may write

$$(158) \quad \begin{aligned} \int_0^{t'} \varphi(n, -1, t)dt &= -\tfrac{1}{2}\sum_{n=0}^{n}(2n+1)(-1)^n \int_{\pi}^{\theta} X_n(\cos\theta)\sin\theta d\theta \\ &= -\tfrac{1}{2}\sum_{n=0}^{n}(2n+1)\int_{\pi}^{\theta} X_n\{\cos(\pi-\theta)\}\sin\theta d\theta \\ &= -\tfrac{1}{2}\int_{\pi}^{\theta} Y_n'\{\cos(\pi-\theta)\}d\theta \\ &= -\tfrac{1}{2}[Y_n\{\cos(\pi-\theta)\}]_{\theta=\pi}^{\theta} = 1 - \tfrac{1}{2}Y_n\{\cos(\pi-\theta)\}, \end{aligned}$$

from which it appears that for values of θ such that $\eta \leqq \theta \leqq \pi - \eta$ ($\eta > 0$) *i. e.*, of t such that $\epsilon \leqq t \leqq 2 - \epsilon$, the first member of (158) converges uniformly ($n = \infty$) to the limit $+1$, thus satisfying relation $(\mathrm{I})_a$ of § 55 ($G_1 = 1$) when exception is there made of the value $t = 2$ ($\theta = 0$).

Relations $(\mathrm{II})_a$ and $(\mathrm{II})_b$ of § 55 are evidently satisfied as a result of (157) and (158), but relations $(\mathrm{III})_a$ and $(\mathrm{III})_b$ are not satisfied. For example, we have

$$\varphi(n, 1, t) = \tfrac{1}{2}\sum_{n=0}^{n}(2n+1)X_n(1+t) = \tfrac{1}{2}\sum_{n=0}^{n}(2n+1)X_n(\cos\theta)$$

and as n increases indefinitely the right member here appearing becomes an oscillatory divergent series whatever the value of θ.[28] We are here led, therefore, not to an application of remark (1) of § 56, but rather to an entire reconsideration of the reasoning by which Theorems (V) and (VI) of § 55 were established. In this way we may supply conditions for $f(x)$ which, notwithstanding the present exceptional character of $\varphi(n, 1, t)$, will insure the convergence of the series (142) when x is equal to either 1 or -1.

[28] Cf. Frejér, *l. c.*, p. 106.

Thus, if $s_n(1)$ represents the sum of the first $(n+1)$ terms of the series (142) when $x = 1$, we may write

$$(159)\quad s_n(1) = \int_{-\epsilon}^{0} f(1+t)\,\varphi(n, 1, t)dt + \int_{-2}^{-\epsilon} f(1+t)\varphi(n, 1, t)dt; \qquad \epsilon > 0.$$

Since, as already shown, relations $(\mathrm{I})_b$ and $(\mathrm{II})_b$ are satisfied, it follows (cf. (25)) that the first term here appearing on the right approaches the limit $f(1+0)$ provided only that the integral

$$\int_{-1}^{1} |f(x)|\,dx$$

exists and that $f(x)$ has limited total fluctuation in the neighborhood at the left of the point $x = 1$. It remains, therefore, but to impose such further conditions upon $f(x)$ that the last term of (159) shall approach the limit zero as $n = \infty$, and we shall now show that this will be the case whenever $f(x)$ is of limited total fluctuation throughout the whole interval $(-1, 1)$.

First, let us consider the integral

$$(160)\qquad \int_{-2+\epsilon}^{-\epsilon} f(1+t)\varphi(n, 1, t)dt.$$

Considering that n has any fixed value (positive integral), let us divide the interval $(-2+\epsilon, -\epsilon)$ into a certain number m of parts such that in each the function $\varphi(n, 1, t)$ does not change sign. Let $p_1, p_2, \cdots, p_{m-1}$ be the corresponding points of division. We may then write

$$\int_{-2+\epsilon}^{-\epsilon} f(1+t)\varphi(n, 1, t)dt = \left(\int_{-2+\epsilon}^{p_1} + \int_{p_1}^{p_2} + \cdots + \int_{p_{m-1}}^{-\epsilon}\right) f(1+t)\varphi(n, 1, t)dt$$

$$= f_1\int_{-2+\epsilon}^{p_1} \varphi dt + f_2\int_{p_1}^{p_2} \varphi dt + \cdots + f_{m-1}\int_{p_{m-2}}^{p_{m-1}} \varphi dt + f_m\int_{p_{m-1}}^{-\epsilon} \varphi dt,$$

where $\varphi = \varphi(n, 1, t)$ and $f_1, f_2, f_3, \cdots, f_m$ are certain values lying between the upper and lower limits of $f(1+t)$ when considered within the intervals

$$(-2+\epsilon, p_1), \qquad (p_1, p_2), \qquad \cdots, \qquad (p_{m-2}, p_{m-1}), \qquad (p_{m-1}, -\epsilon)$$

respectively. Whence, if we let $\theta_1, \theta_2, \cdots, \theta_{m-1}, \theta_m$ be the values of

$$(161)\qquad \int_{-2+\epsilon}^{t} \varphi(n, 1, t)dt$$

at the points $t = p_1, t = p_2, \cdots, t = p_m$ respectively, we may write

$$\int_{-2+\epsilon}^{-\epsilon} f(1+t)\varphi(n, 1, t)dt = \theta_1(f_1 - f_2) + \theta_2(f_2 - f_3) + \cdots$$
$$+ \theta_{m-1}(f_{m-1} - f_m) + \theta_m f_m.$$

Since, as already shown, the integral (157) when considered for values of t in the interval $-2+\epsilon \leqq t \leqq -\epsilon$ converges uniformly ($n=\infty$) to the limit -1, it follows that the integral (160), when considered for the same values of t, converges uniformly to the limit zero. Whence, if σ be a preassigned arbitrarily small positive quantity we shall have, at least if n be chosen sufficiently large, $|\theta_1|<\sigma$, $|\theta_2|<\sigma$, $\cdots$, $|\theta_m|<\sigma$. Whence, also, if λ represents the upper limit of $f(1+t)$ between $t=-2+\epsilon$ and $t=-\epsilon$, and if D_s represents the fluctuation of $f(1+t)$ in the interval $p_s<t<p_{s+1}$, the last equation enables us to write

$$\left|\int_{-2+\epsilon}^{-\epsilon} f(1+t)\varphi(n,1,t)dt\right| < \sigma\left(\lambda+\sum_{s=1}^{m-1} D_s\right),$$

from which the indicated result concerning the last term of (159) becomes evident.

Similarly, when $x=-1$ we may obtain the corresponding result so that the discussion of the convergence of the series (142) may now be readily completed, both for the case of a point x such that $-1<x<1$ and for the end points $x=\pm 1$, except that, following remark (1) of § 56, it remains to consider the integrals

$$(162)\qquad \begin{aligned} &\int_{-1-\alpha}^{-1-\alpha+\xi} f(\alpha+t)\varphi(n,\alpha,t)dt, &&\int_{1-\alpha-\xi}^{1-\alpha} f(\alpha+t)\varphi(n,\alpha,t)dt,\\ &\int_{-2}^{-2+\xi} f(1+t)\varphi(n,1,t)dt, &&\int_{2-\xi}^{2} f(-1+t)\varphi(n,-1,t)dt. \end{aligned}$$

In order to complete the discussion it thus suffices to show that, at least if ξ be taken sufficiently small, each of these integrals remains less in absolute value than any preassigned positive quantity ω provided n be greater than some fixed quantity N. Moreover, in the case of the first two integrals, this property should be true uniformly for all values of α such that $-1<a'\leqq\alpha\leqq b'<1$—*i. e.*, the determination of N should not depend upon α.

Taking the first of the integrals (162), let us now suppose that $f(x)$ is of limited total fluctuation in the neighborhood at the right of the point $x=-1$ and hence that $f(\alpha+t)$ has the same property at the right of the point $t=-1-\alpha$. Then, since we have already shown that the integral (143) converges ($n=\infty$) to the limit $-\frac{1}{2}$ and that the convergence is uniform for all values of α and t such that $-1<a'\leqq\alpha\leqq b'<1$; $-1-\alpha\leqq t\leqq-\epsilon$, it follows that we may treat the first of the integrals (162) in the same manner as we treated the integral (160), thus showing that however small the choice of the positive quantity σ, we may determine a value N (dependent only on σ) such that

$$\left|\int_{-1-\alpha}^{-1-\alpha+\xi} f(\alpha+t)\varphi(n,\alpha,t)dt\right| < \sigma\left(\lambda'+\sum_{s=1}^{m-1} D_s'\right),$$

where λ' is the upper limit of $f(\alpha + t)$ in the interval $-1 - \alpha < t < -1 - \alpha + \xi$ and where

$$\sum_{s=1}^{m-1} D_s'$$

represents the sum of the oscillations of $f(\alpha + t)$ corresponding to a certain division of the same interval—a sum which by hypothesis is less than a constant.

Similarly, the second of the integrals (162) is found to have the properties desired.

As regards the third integral, the method just employed cannot here be used because we have not investigated the convergence of the integral (157) when $-2 \leqq t \leqq -2 + \xi$. We may, however, show as follows that if $f(x)$ is assumed to be of limited total fluctuation in the neighborhood at the right of the point $x = -1$ (as already implied in the conditions which we have placed upon $f(x)$ in order that relation $(\mathrm{III})_b$ be satisfied) then the third integral of (162) has the properties desired. In fact, following again remark (1) of § 56, we may then show that the integral in question may be made less in absolute value than any preassigned positive quantity by taking ξ sufficiently small, this being true uniformly for all values of n sufficiently large. To see this, if we let the accent denote differentiation with respect to θ, we have from (157)

$$(163) \qquad \int_{-2}^{-2+\xi} f(1 + t)\varphi(n, 1, t)dt = -\tfrac{1}{2}\int_{\pi-\eta}^{\pi} f(\cos\theta)Y'(\cos\theta)d\theta,$$

where η depends only upon ξ and vanishes when $\xi = 0$. Since $f(x)$ has been assumed to be of limited total fluctuation in the neighborhood at the right of the point $x = -1$, the same function is either monotone in this interval or consists of the sum of a finite number of such functions.[29] Evidently, then, without loss of generality we may assume in the study of (163) that $f(\cos\theta)$ is monotone in the interval $\pi - \eta < \theta < \pi$.

This being the case, let us apply the second law of the mean for integrals to the second member of (163). We obtain

$$\int_{-2}^{-2+\xi} f(1 + t)\varphi(n, 1, t)dt = -\tfrac{1}{2}f(-1 + \xi)\int_{\pi-\eta}^{\pi-\eta_1} Y_n'(\cos\theta)d\theta$$
$$-\tfrac{1}{2}f(-1 + 0)\int_{\pi-\eta_1}^{\pi} Y_n'(\cos\theta)d\theta,$$

which, upon recalling that $Y_n(\cos\pi) = 0$, reduces to

$$-\tfrac{1}{2}\{f(-1 + \xi) - f(-1 + 0)\}Y_n(\cos(\pi - \eta_1)) + \tfrac{1}{2}f(-1 + \xi)Y_n(\cos(\pi - \eta)),$$

and of the two terms here appearing, the first, upon recalling that

$$|Y_n(\cos(\pi - \eta_1))| \leqq 2,$$

[29] Cf. § 46, p. 110.

may be made arbitrarily small in absolute value by choosing ξ sufficiently small, while the second (ξ having been fixed) vanishes as $n = \infty$, it being observed here that η does not depend upon n, so that we are dealing with the expression $Y_n(\cos\theta)$, wherein θ has a fixed value such that $0 < \theta < \pi$.

Similarly, it appears that the last of the integrals (162) has the properties desired in case $f(x)$ is assumed to be of limited total fluctuation in the neighborhood at the right of the point $x = 1$.

In summary, then, we reach the following theorem respecting the convergence of the series (142):

THEOREM I. *If the function $f(x)$ of the real variable x satisfies the following three conditions:*

(*a*) *remains finite throughout the interval* $(-1, 1)$ *with the possible exception of a finite number of points;*

(*b*) *is such that the integral*

$$\int_{-1}^{1} |f(x)|\,dx$$

exists;

(*c*) *is of limited total fluctuation in an arbitrarily small neighborhood at the right of the point $x = -1$ and in a similar neighborhood at the left of the point $x = 1$, then the series*

$$(164)\qquad \sum_{n=0}^{\infty} q_n X_n(x); \qquad q_n = \frac{2n+1}{2}\int_{-1}^{1} f(x) X_n(x)\,dx,$$

in which $X_n(x)$ represents the polynomial of Legendre (Zonal Harmonic) of order n, will converge at any point x $(-1 < x < 1)$ in the arbitrarily small neighborhood of which $f(x)$ has limited total fluctuation, and the sum will be

$$\tfrac{1}{2}[f(x-0) + f(x+0)].$$

Moreover, the convergence will be uniform (§ 45) to the limit $f(x)$ throughout any interval (a', b') enclosed within a second interval (a_1, b_1) such that $-1 < a_1 < a' < b' < b_1 < 1$ provided that $f(x)$ is continuous throughout (a', b') inclusive of its end points and has limited total fluctuation throughout (a_1, b_1).

Also, if we replace conditions (a), (b) and (c) by the single more restrictive condition; viz., that $f(x)$ be of limited total fluctuation throughout the whole interval $(-1, 1)$ then the same series will converge when $x = -1$ or $x = 1$ and the respective sums will be $f(-1+0)$, $f(1-0)$.[30]

[30] The results contained in this theorem, both for the case of an internal point $(-1 < x < 1)$ and for that of the end points $x = \pm 1$, appear to have been first established rigorously by HOBSON (*Proc. London Math. Soc.*, Vol. 6 (1908), p. 395. *Ibid.*, Vol. 7 (1909), p. 39). DINI's consideration of the problem ("Serie di Fourier, etc.," pp. 278–282), although outlining all the essential steps of the required analysis, is but fragmentary, especially that which concerns the end points. In HOBSON's second paper, just noted, less stringent conditions for $f(x)$ are obtained than those of the Theorem above, the same resulting from an extended critical study of the behavior of $X_n(x)$, $(-1 \leqq x \leqq 1)$ for large values of n (*l. c.*, pp. 25–30).

69. We proceed to consider the summability ($r = 1$) of the series (142) and in so doing we shall make use of the following well known result:[31]

"If we place

$$s_n(\gamma) = \sum_{n=0}^{n} (2n+1) X_n(\cos\gamma) \tag{165}$$

and

$$s_n'(\gamma) = \frac{1}{n+1}[s_0(\gamma) + s_1(\gamma) + \cdots + s_n(\gamma)] \tag{166}$$

then for large values of n we have

$$s_n'(\gamma) = \frac{1}{\sqrt{n}}\left\{\frac{4\cos\frac{\gamma}{2}}{\sqrt{\pi}\sin\frac{\gamma}{2}(2\sin\gamma)^{\frac{3}{2}}}\cos\left[\left(n+\frac{3}{2}\right)\gamma - \frac{5\pi}{4}\right] + \delta_n'(\gamma)\right\}, \tag{167}$$

where $\lim_{n=\infty} \delta_n'(\gamma) = 0$ uniformly for all $\epsilon \leqq \gamma \leqq \pi - \epsilon$ ($\epsilon > 0$)."

It will thus appear that although the summability ($r = 1$) of (142) at an internal point ($-1 < x < 1$) cannot be assured under conditions so slightly limitive as those met with in the corresponding studies of Fourier series (§ 46) or the Bessel expansions (§ 67), nor indeed under restrictions upon $f(x)$ which are any less than those stated in Theorem I for convergence ($r = 0$) at such a point, yet at the end points $x = \pm 1$ the conditions for summability may be stated in a less restrictive form than the corresponding ones in Theorem I.

We begin by noting that, as a result of (144) and (145), the function $\Phi(\alpha, n, t)$ corresponding to the present development is such that

$$\Phi(n, \alpha, t) = \frac{1}{n+1}[\varphi(n, \alpha, t) + \varphi(n-1, \alpha, t) + \cdots + \varphi(0, \alpha, t)],$$

where

$$\varphi(n, \alpha, t) = \frac{1}{4\pi}\int_0^{2\pi} \sum_{n=0}^{n} (2n+1) X_n(\cos\gamma)\, d\varphi$$

the angle γ being here determined, as in § 68, through the following relations: $\alpha = \cos\theta'$, $\quad \alpha + t = \cos\theta$, $\quad \cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi')$, it being understood that φ' is assigned any fixed value ($0 \leqq \varphi' < 2\pi$) independent of θ.

Thus we may write

$$\Phi(n, \alpha, t) = \frac{1}{4\pi}\int_0^{2\pi} s_n'(\gamma)\, d\varphi, \tag{168}$$

where $s_n'(\gamma)$ is defined as in (166).

Now, when t is such that $-\epsilon \leqq t \leqq \epsilon$ (as occurs in relation (II) of the general theorem of § 52) the corresponding values of γ pertaining to the neighborhood

[31] Cf. FEJÉR, *l. c.*, p. 107.

of the (fixed) point (θ', φ') lie in an interval of the form $0 \leqq \gamma \leqq \eta$ where η vanishes with ϵ. Thus, while formula (168) is general, holding for all values of α and t with which we are concerned in applying the theorem of § 52 to the present development, we are unable to determine whether relation (II)′ of the same theorem is here satisfied until more than is given by (177) is known of the behavior of $s_n'(\gamma)$ for large values of n. A critical study of $s_n'(\gamma)$ for $0 \leqq \gamma \leqq \epsilon$ is here needed and such study has apparently not yet been made.

Again, it cannot be argued from (167) and (168) that relation (III) of § 51 is here satisfied by $\Phi(n, \alpha, t)$ (cf. remark (2), § 56). This relation, however, is seen to be satisfied if we confine ourselves to the intervals $-1-\alpha+\xi \leqq t \leqq -\epsilon$, $\epsilon \leqq t \leqq 1-\alpha-\xi$ $(\xi > 0)$ instead of $-1-\alpha \leqq t \leqq -\epsilon$, $\epsilon \leqq t \leqq 1-\alpha$, but this is nothing more than can be at once inferred from the properties already pointed out in § 68 regarding the present function $\varphi(n, \alpha, t)$. It may be noted that if we could show that $s_n'(\gamma)$ when considered for all values of n and for values of γ within the interval $\pi - \epsilon \leqq \gamma \leqq \pi$ remains less than a constant (dependent only upon ϵ) the function $\Phi(n, \alpha, t)$ would come to completely satisfy relation (III) as a result of (167) and (168). That such is true of $s_n'(\gamma)$ seems probable.

The conclusion from these remarks respecting summability $(r = 1)$ at an internal point x $(-1 < x < 1)$ is therefore purely negative, except naturally that such summability will necessarily be present[32] under the conditions for convergence $(r = 0)$ as given in the theorem of the preceding §.[33]

Turning to a consideration of the summability $(r = 1)$ of the series (164) when $x = -1$ or $x = 1$, we see upon reference to the results obtained for $\varphi(n, -1, t)$ and $\varphi(n, 1, t)$ in § 68 that relations $(\mathrm{I})_a$, $(\mathrm{II})_a$, $(\mathrm{I})_b$ and $(\mathrm{II})_b$ of § 55 are satisfied by the present functions $\Phi(n, -1, t)$ and $\Phi(n, 1, t)$ (regarded as functions of the type φ there indicated) except that doubt exists in the case of $(\mathrm{I})_a$ and $(\mathrm{I})_b$ when t belongs to the respective intervals $2 - \xi \leqq t \leqq 2$, $-2 \leqq t \leqq -2 + \xi$ $(\xi > 0)$. In other words, nothing more can be said of $\Phi(n, -1, t)$ and $\Phi(n, 1, t)$ than was said of $\varphi(n, -1, t)$ and $\varphi(n, 1, t)$ in § 68. This, however, is not the case in dealing with relations $(\mathrm{III})_a$ and $(\mathrm{III})_b$.

Thus, in $(\mathrm{III})_b$ we have to consider the expression

$$\Phi(n, 1, t) = \frac{1}{n+1}[\varphi(n, 1, t) + \varphi(n-1, 1, t) + \cdots + \varphi(0, 1, t)],$$

where

$$\varphi(n, 1, t) = \tfrac{1}{2}\sum_{n=0}^{n}(2n+1)X_n(\cos\theta) = \tfrac{1}{2}s_n(\theta).$$

We may therefore write

(169) $$\Phi(n, 1, t) = \tfrac{1}{2}s_n'(\theta),$$

[32] Cf. § 44.

[33] Cf. Chapman, *Quart. Journ. Math.*, Vol. 43 (1911), p. 51. For summability $(r = 1)$ Chapman places no restrictions upon $f(x)$ at the extremities of the interval $(-1 < x < 1)$ other than those for the *whole* interval.

so that upon introducing (167) we see that $(\mathrm{III})_b$ is here satisfied for all values of t in the interval $-2+\xi \leqq t \leqq -\epsilon$. For the remaining values of t with which $(\mathrm{III})_b$ is concerned, *i. e.*, $-2 \leqq t \leqq -2+\xi$, doubt exists.

Likewise, relation $(\mathrm{III})_a$ is seen to be satisfied by $\Phi(n, -1, t)$ except possibly for values of t in the interval $2-\xi \leqq t \leqq 2$.

From the general theorem of § 52 together with the remarks in § 56 and the investigations already made in § 68 of the last two of the integrals (162) we reach the following

THEOREM II. *If the function $f(x)$ of the real variable x satisfies conditions* (*a*), (*b*) *and* (*c*) *of the Theorem I* (§ 68) *then the series* (164) *when considered for the values $x = \pm 1$ will be summable* $(r = 1)$ *to the respective limits* $f(1-0)$, $f(-1+0)$.

70. The difficulties which present themselves in the study of the summability, $r = 1$, of the series (164) disappear in large measure when we consider the same problem with $r = 2$. This fact was first pointed out by FEJÉR[34] who confined himself, however, to functions $f(x)$ having somewhat greater limitations than we shall here find necessary in view of the general theorems of § 52. In what follows we shall make use without further remark of the following two preliminary results which may be found established on pages 81–87 of FEJÉR's original memoir.

"Having defined $s_n(\gamma)$ and $s_n'(\gamma)$ as in (165) and (166), if we place[35]

$$(170) \qquad s_n''(\gamma) = \frac{1}{n+1}[s_0'(\gamma) + s_1'(\gamma) + \cdots + s_n'(\gamma)]$$

then

"(1) Whatever the values of n and γ $(0 < \gamma < \pi)$, $s_n''(\gamma)$ is never negative.

"(2) For values of γ such that $\epsilon \leqq \gamma \leqq \pi$, ϵ being arbitrarily small but > 0, the expression $s_n''(\gamma)$ converges $(n = \infty)$ uniformly to zero."

These results being premised, we shall now endeavor to apply the general theorem of § 52 to the present development.

Just as we found the formula (168) for the function $\Phi(n, \alpha, t)$ arising in the study of the summability, $r = 1$, so it appears that if we represent by $\psi(n, \alpha, t)$ the corresponding function which arises when $r = 2$, we shall have

$$(171) \qquad \psi(n, \alpha, t) = \frac{1}{4\pi}\int_0^{2\pi} s_n''(\gamma)d\varphi,$$

where $s_n''(\gamma)$ is given by (170). Whence, upon using result (1) above, we see that

$$\int_{-\epsilon}^{\epsilon} |\psi(n, \alpha, t)|dt = -\int_{-\epsilon}^{\epsilon} \psi(n, \alpha, t)dt.$$

Thus, in view of the fact that the function $\varphi(n, \alpha, t)$ (cf. (144)) and hence

[34] Cf. *Math. Annalen,* Vol. 67 (1909), pp. 76–109.

[35] Thus, $s_n''(\gamma)$ comes to represent Hölder's second mean for the series (164).

$\psi(n, \alpha, t)$ satisfies relation (II) of the theorem of § 51 (as shown in § 68) it follows that the present function $\psi(n, \alpha, t)$ satisfies relation (II)$'$ of § 52.

Moreover, if we avail ourselves of result (2) above, it appears from (171) that $\psi(n, \alpha, t)$, when regarded as one of the functions of the type $\varphi(n, \alpha, t)$ of § 51, satisfies relation (III) of the same §.

It remains but to note that $\psi(n, \alpha, t)$, when considered as one of the functions $\varphi(n, \alpha, t)$ of § 51 satisfies relation (I) of that §, as a result of our analysis in § 68 in order to see that the conditions for the use of the general theorem of § 51 are here all satisfied.

As regards the summability ($r = 2$) of the series (164) when $x = \pm 1$, it is easily seen that $\psi(n, 1, t)$ and $\psi(n, -1, t)$ satisfy respectively all the conditions demanded by the general theorems VII and VIII of § 55. Thus, upon referring to (157) and recalling that $Y_n(\cos\theta) = X_{n+1}(\cos\theta) + X_n(\cos\theta)$, we have but to make use of result (2) above to see that $\psi(n, 1, t)$ satisfies relation (I)$_b$ ($a = -1$, $b = 1$, $G_2 = 1$) of Theorem VI (§ 55). Relation (II)$_b'$ of Theorem VIII (§ 55) is also satisfied, as follows from the fact established in § 68 that $\varphi(n, 1, t)$ satisfies (II)$_b$, Theorem VI (§ 55), while from result (1) above, we may write

$$\int_{-\epsilon}^{0} |\psi(n, 1, t)|\,dt = -\int_{-\epsilon}^{0} \psi(n, 1, t)\,dt.$$

Finally, it follows from (169) that $\psi(n, 1, t) = \frac{1}{2}s_n''(\theta)$ so that by applying result (2) above we see that $\psi(n, 1, t)$ satisfies relation (III)$_b$, Theorem VI (§ 55).

Upon noting the corresponding results concerning $\psi(n, -1, t)$ and applying the Theorems of § 55 we reach in summary the following

THEOREM III. *If $f(x)$ be any function which, when considered throughout the interval $(-1, 1)$, satisfies conditions (A) and (B) of Theorem I (§ 51) then the series (164) will be summable ($r = 2$) at any point x ($-1 < x < 1$) for which the limits $f(x - 0)$, $f(x + 0)$ exist and the sum will be*

$$\tfrac{1}{2}\,[f(x - 0) + f(x + 0)].$$

Moreover, the summability will be uniform (§ 45) to the limit $f(x)$ throughout any interval (a', b') such that $-1 < a' < b' < 1$ provided that at all points within (a', b') inclusive of the end points the function $f(x)$ is continuous.

Under the same conditions (A), (B) the series when considered for the values $x = -1$ and $x = 1$ will be summable ($r = 2$) to the respective limits $f(-1 + 0)$, $f(1 - 0)$ provided only that these limits exist.[36]

[36] Interesting results have been obtained by PLANCHEREL (*Rend. del Cir. Mat. di Palermo*, Vol. 33 (1912), pp. 41–66) relative to the summability of the Legendre developments when, instead of adopting the Hölder definition of sum, one employs that of De la Vallée Poussin (see footnote, p. 77).

APPENDIX

1. *Proof of statement* (I), § 46. It is desirable for the purpose to establish the following two lemmas:

Lemma I. If a and b are any two real numbers such that either $-\pi + \epsilon \leqq b < a \leqq -\epsilon$ *or* $\epsilon \leqq a < b \leqq \pi - \epsilon$, ϵ *being an arbitrarily small positive quantity, and if k is a positive quantity which may increase indefinitely according to any law whatever, then*

$$\lim_{k=\infty} \int_a^b \frac{\sin kt}{\sin t}\, dt = 0. \tag{1}$$

and the limit is approached uniformly for all the indicated values of a and b.

In order to establish this, let us suppose first that a and b are positive and divide the cases which may then arise into three sets as follows:

$$(a)\ a < b \leqq \frac{\pi}{2}; \qquad (b)\ a < \frac{\pi}{2} < b; \qquad (c)\ \frac{\pi}{2} \leqq a < b.$$

In (a) we have merely to note that as t varies from a to b the function $1/\sin t$ is always positive and continually decreasing so that we may apply the second law of the mean for integrals and write

$$\int_a^b \frac{\sin kt}{\sin t}\, dt = \frac{1}{\sin a} \int_a^\xi \sin kt dt = \frac{1}{\sin a}\left[\frac{\cos ka - \cos k\xi}{k}\right],$$

where ξ is a certain quantity lying between a and b. Hence, in (a) we shall have

$$\left| \int_a^b \frac{\sin kt}{\sin t}\, dt \right| \leqq \frac{2}{k \sin a} \leqq \frac{2}{k \sin \epsilon}, \tag{2}$$

from which the indicated result becomes evident.

In (b) we write

$$\int_a^b \frac{\sin kt}{\sin t}\, dt = \int_a^{\pi/2} \frac{\sin kt}{\sin t}\, dt + \int_{\pi/2}^b \frac{\sin kt}{\sin t}\, dt, \tag{3}$$

where the first integral of the second member falls in group (a), while the last one, after making the substitution $t = \pi - t'$, may be written

$$\int_{\pi-b}^{\pi/2} \frac{\sin k(\pi - t)}{\sin t}\, dt.$$

In this integral as t varies from $\pi - b$ to $\pi/2$ the function $1/\sin t$ is always positive and continually decreasing so that we may again apply the second law of the mean and write

$$\int_{\pi-b}^{\pi/2} \frac{\sin k(\pi - t)}{\sin t}\, dt = \frac{1}{\sin b} \int_{\pi-b}^{\xi} \sin k(\pi - t) dt = \frac{1}{\sin b}\left[\frac{\cos kb - \cos k(\pi - \xi)}{k}\right],$$

where $\pi - b < \xi < \pi/2$.

Whence,

$$\left| \int_{\pi-b}^{\pi/2} \frac{\sin k(\pi - t)}{\sin t}\, dt \right| < \frac{2}{k \sin \epsilon},$$

after which the indicated result becomes evident as before.

In (c) we have, after making the substitution $t = \pi - t'$,

$$\int^b \frac{\sin kt}{\ } dt = \int^{\pi-a} \frac{\sin k(\pi - t)}{\ } dt$$

Upon noting that the absolute value of the expression (1) remains unchanged when $-a$ and $-b$ are substituted for a and b respectively, the Lemma thus becomes completely established.

Lemma II. If k be a positive quantity increasing according to any law whatever and if b be a constant (independent of k) and such that $0 < \epsilon \leqq b \leqq \pi - \epsilon$, ϵ being an arbitrarily small positive quantity, then

$$\lim_{k=\infty} \int_0^b \frac{\sin kt}{\sin t}\, dt = \frac{\pi}{2},[1]$$

and the limit is approached uniformly for all the indicated values of b.

In order to prove this let us indicate by k' the first odd number equal to or greater than k and let us place $k = k' - \gamma$ so that $0 \leqq \gamma \leqq 2$.

We may then write

$$\int_0^b \frac{\sin kt}{\sin t}\, dt = \int_0^\epsilon \frac{\sin kt}{\sin t}\, dt + \int_\epsilon^b \frac{\sin kt}{\sin t}\, dt,$$

in which the last term, by reason of Lemma I, approaches uniformly the limit zero as $k = \infty$.

Also, we may write

$$\int_0^\epsilon \frac{\sin kt}{\sin t}\, dt = \int_0^\epsilon \frac{\sin (k' - \gamma)t}{\sin t}\, dt = \int_0^\epsilon \frac{\sin k't \cos \gamma t}{\sin t}\, dt - \int_0^\epsilon \frac{\cos k't \sin \gamma t}{\sin t}\, dt,$$

and by reason of the general formula $f(\delta) = f(0) + \delta f'(\theta\delta)$; $0 < \theta < 1$, we may place

$$\cos \gamma t = 1 - \gamma t \sin \gamma_1 t$$

where $0 \leqq \gamma_1 \leqq \gamma < 2$ and hence

$$\int_0^\epsilon \frac{\sin kt}{\sin t}\, dt = \int_0^\epsilon \frac{\sin k't}{\sin t}\, dt - \int_0^\epsilon \frac{\gamma t \sin k't \sin \gamma_1 t}{\sin t}\, dt - \int_0^\epsilon \frac{\cos k't \sin \gamma t}{\sin t}\, dt. \tag{4}$$

Of the three integrals last appearing on the right, the first may be written in the form

$$\left(\int_0^{\pi/2} - \int_\epsilon^{\pi/2}\right) \frac{\sin k't}{\sin t}\, dt = \frac{\pi}{2} - \int_\epsilon^{\pi/2} \frac{\sin k't}{\sin t}\, dt, \tag{5}$$

since, if n be the integer such that $k' = 2n + 1$, we have

$$\int_0^{\pi/2} \frac{\sin k't}{\sin t}\, dt = \int_0^{\pi/2} \left[1 + \sum_{n=1}^{n} \cos 2nt\right] dt = \frac{\pi}{2}.$$

Upon applying Lemma I to the last integral of (5) it thus appears that the integral of (4) in question approaches the limit $\pi/2$ as $k = \infty$.

As regards the last two integrals of (4), it is at once evident that each of these may be made arbitrarily small in absolute value with ϵ and with this the proof of the Lemma becomes complete.

The proof of (I) of § 46 may now be made as follows:

We may write

$$\int_0^t \varphi(n, t)dt = -\frac{1}{2\pi}\left(\int_t^{-\epsilon} + \int_{-\epsilon}^{0}\right) \frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}}\, dt$$

$$= -\frac{1}{\pi} \int_{t/2}^{-\epsilon/2} \frac{\sin (2n+1)t}{\sin t}\, dt - \frac{1}{\pi} \int_0^{\epsilon/2} \frac{\sin (2n+1)t}{\sin t}\, dt$$

and when $-2\pi + \epsilon < t < -\epsilon$ we have $-\pi + \epsilon/2 < t/2 < -\epsilon/2$ so that the first term here appearing in the last member approaches uniformly the limit zero when $n = \infty$, as appears from Lemma I. The last term of the same member, however, approaches the limit $-\frac{1}{2}$ as follows from Lemma II.

Similarly, when $\epsilon < t < 2\pi - \epsilon$ the desired result follows directly from Lemmas I and II upon writing

[1] Cf. Dini, Serie di Fourier, etc., § 19.

$$\int_0^t \varphi(n, t)dt = \frac{1}{2\pi}\left(\int_\epsilon^t + \int_0^\epsilon\right)\frac{\sin\frac{2n+1}{2}t}{\sin\frac{t}{2}}\,dt = \frac{1}{\pi}\int_{\epsilon/2}^{t/2}\frac{\sin(2n+1)t}{\sin t}\,dt + \frac{1}{\pi}\int_0^{\epsilon/2}\frac{\sin(2n+1)t}{\sin t}\,dt.$$

2. *Proof of statement* (II) *of* § 46. We first establish the following Lemma:

Lemma III. If k is a positive quantity which may increase indefinitely according to any law whatever we may write[2] *for any value however great of k and for any value of t such that* $-\pi/2 \leqq t \leqq \pi/2$:

$$\left|\int_0^t \frac{\sin kt}{\sin t}\,dt\right| < \pi.$$

In fact, considering that k has any particular value among those which it may take and considering first the cases in which t is positive, we observe that since the function $\sin kt$ vanishes by changing sign at the points π/k, $2\pi/k$, $3\pi/k$, $\cdots$, while the integral

$$\int_0^t \frac{\sin kt}{\sin t}\,dt \tag{6}$$

is positive from $t = 0$ to $t = \pi/k$, this integral has maximum values at the points π/k, $3\pi/k$, $5\pi/k$, $\cdots$ and minimum values at the points $2\pi/k$, $4\pi/k$, $\cdots$.

Moreover, the greatest of these maximum values is

$$\int_0^{\pi/k} \frac{\sin kt}{\sin t}\,dt, \tag{7}$$

for we may show as follows that the difference between any maximum value and the next succeeding one is positive:

Let

$$\sigma_1 = \frac{(2s+1)\pi}{k} \quad\text{and}\quad \sigma_2 = \frac{(2s+3)\pi}{k} \qquad \left(s = 0, 1, 2, 3, \cdots \quad\text{and}\quad \frac{(2s+3)\pi}{k} \leqq \frac{\pi}{2}\right)$$

be any two successive points belonging to the set π/k, $3\pi/k$, $5\pi/k$, $\cdots$. The difference between the corresponding maximum values of the integral in question is

$$\left(\int_0^{\sigma_1} - \int_0^{\sigma_2}\right)\frac{\sin kt}{\sin t}\,dt = -\int_{\sigma_1}^{\sigma_2}\frac{\sin kt}{\sin t}\,dt = -\frac{1}{k}\int_{k\sigma_1}^{k\sigma_2}\frac{\sin t}{\sin\frac{t}{k}}\,dt =$$

$$-\frac{1}{k}\int_{(2s+1)\pi}^{(2s+2)\pi}\frac{\sin t}{\sin\frac{t}{k}}\,dt + \frac{1}{k}\int_{(2s+1)\pi}^{(2s+2)\pi}\frac{\sin t}{\sin\frac{t+\pi}{k}}\,dt = -\frac{1}{k}\int_{(2s+1)\pi}^{(2s+2)\pi}\sin t\left[\frac{1}{\sin\frac{t}{k}} - \frac{1}{\sin\frac{t+\pi}{k}}\right]dt.$$

In the last integral here appearing the factor $\sin t$ is negative (or zero) for all the values of t between $(2s+1)\pi$ and $(2s+2)\pi$ and since, for the same values of t, we have

$$0 < \frac{t}{k} < \frac{t+\pi}{k} \leqq \frac{\pi}{2},$$

the factor appearing in square brackets in the last integral is positive when

$$(2s+1)\pi \leqq t \leqq (2s+2)\pi.$$

In like manner it appears that the least of the minimum values of (6) is

$$\int_0^{2\pi/k}\frac{\sin kt}{\sin t}\,dt$$

and that this value is positive together with all values of the integral (6) when $0 < t \leqq 2\pi/k$.

Thus, for all values of t such that $0 < t \leqq \pi/2$ the integral (6) is positive and in summary we may say that the greatest absolute value of (6) when $0 \leqq t \leqq \pi/2$ is given by (7). But

[2] Cf. Dini, l. c., § 18.

$$\int_0^{\pi/k} \frac{\sin kt}{\sin t}\,dt = \frac{\sin k_1 t}{\sin t_1}\int_0^{\pi/k} dt = \pi \frac{\dfrac{\sin kt_1}{kt_1}}{\dfrac{\sin t_1}{t_1}}\,;\ 0 < t_1 < \frac{\pi}{k}$$

and since for such values of t_1 this expression is < 1 the Lemma now follows provided $t \geqq 0$. In order to prove it also for the cases in which $t < 0$ we need but to note that

$$\left| \int_0^{-t} \frac{\sin kt}{\sin t}\,dt \right| = \left| \int_0^{t} \frac{\sin kt}{\sin t}\,dt \right|.$$

By use of Lemma III the proof of (II), § 46, is immediate since we have

$$\left| \int_0^t \varphi(n, t)\,dt \right| = \frac{1}{2\pi}\left| \int_0^t \frac{\sin \dfrac{2n+1}{2} t}{\sin \dfrac{t}{2}}\,dt \right| = \frac{1}{\pi}\left| \int_0^{t/2} \frac{\sin (2n+1)t}{\sin t}\,dt \right| < 1.$$

A possible choice of the constant A of (II) is therefore $A = 1$.

3. *Proof of Statement* (II)′, § 47. We shall here establish the following general lemma:

Lemma IV. If $k = n\alpha + \beta$ where n takes only positive integral values and α and β are any two constants (independent of n) of which $\alpha > 0$, then, corresponding to any $\epsilon > 0$ such that $\epsilon < \pi/2$, $\epsilon < \pi/\alpha$, we shall have for all values of n sufficiently large

$$(8)\qquad \frac{1}{n}\int_{-\epsilon}^{\epsilon} \left| \sum_{n=0}^{n} \frac{\sin kt}{\sin t} \right| dt < g,$$

where g is a certain constant independent of both n and ϵ.

Since

$$\frac{\sin (-kt)}{\sin (-t)} = \frac{\sin kt}{\sin t},$$

it will evidently suffice to prove the lemma for the expression

$$(9)\qquad \frac{1}{n}\int_0^{\epsilon} \left| \sum_{n=0}^{n} \frac{\sin kt}{\sin t} \right| dt$$

instead of (8).

Now, we have

$$\frac{\sin kt}{\sin t} = \frac{1}{\sin t}[\sin n\alpha t \cos \beta t + \cos n\alpha t \sin \beta t],$$

so that by application of the well known formulae

$$(10)\qquad \sum_{n=0}^{n} \sin nx = \frac{\sin \dfrac{nx}{2} \sin (n+1)\dfrac{x}{2}}{\sin \dfrac{x}{2}},$$

$$(11)\qquad \sum_{n=0}^{n} \cos nx = \frac{\cos \dfrac{nx}{2} \sin (n+1)\dfrac{x}{2}}{\sin \dfrac{x}{2}},$$

we obtain

$$(12)\qquad \frac{1}{n}\int_0^{\epsilon} \left| \sum_{n=0}^{n} \frac{\sin kt}{\sin t} \right| dt < \frac{1}{n}\int_0^{\epsilon} \psi_1(n, t)dt + \frac{1}{n}\int_0^{\epsilon} \psi_2(n, t)dt,$$

where

$$\psi_1 = \left| \frac{\sin \dfrac{n\alpha t}{2}}{\sin t} \right| \left| \frac{\sin \dfrac{(n+1)\alpha t}{2}}{\sin \dfrac{\alpha t}{2}} \right|, \qquad \psi_2 = \left| \frac{\sin \dfrac{(n+1)\alpha t}{2}}{\sin t} \right| \left| \frac{\sin \beta t}{\sin \dfrac{\alpha t}{2}} \right|.$$

Now, for the given value of ϵ we may take n so large that

$$\frac{\pi}{(n+1)\alpha} < \epsilon$$

and write

(13) $$\int_0^\epsilon \psi_1 dt = \int_0^{\pi/(n+1)\alpha} \psi_1 dt + \int_{\pi/(n+1)\alpha}^\epsilon \psi_1 dt.$$

Now, when

$$0 < t < \frac{\pi}{(n+1)\alpha} < \epsilon < \frac{\pi}{2}$$

we may write

$$\left|\frac{\sin\frac{n\alpha t}{2}}{\sin t}\right| < \frac{\sin\frac{n\alpha t}{2}}{t} = \frac{n\alpha}{2}\left(\frac{\sin\frac{n\alpha t}{2}}{\frac{n\alpha t}{2}}\right) < \frac{n\alpha}{2}\cdot\frac{\pi}{2} = \frac{\pi n\alpha}{4}$$

and in like manner

(14) $$\left|\frac{\sin\frac{(n+1)\alpha t}{2}}{\sin\frac{\alpha t}{2}}\right| < (n+1)\frac{\sin\frac{(n+1)\alpha t}{2}}{\frac{(n+1)\alpha t}{2}} < (n+1)\frac{\pi}{2}; \qquad 0 < t < \frac{\pi}{(n+1)\alpha}.$$

Thus it appears that the first term on the right in (13) is less than $(\pi^2 n/8)$. Again, the second term on the right in (13) may be put into the form

$$\frac{2}{\alpha}\int_{\pi/(n+1)\alpha}^\epsilon \left|\sin\frac{n\alpha t}{2}\sin\frac{(n+1)\alpha t}{2}\right|\left(\frac{t}{\sin t}\right)\left(\frac{\frac{\alpha t}{2}}{\sin\frac{\alpha t}{2}}\right)\frac{dt}{t^2},$$

which is less than

$$\frac{2}{\alpha}\int_{\pi/(n+1)\alpha}^\epsilon \left(\frac{\pi}{2}\right)^2\frac{dt}{t^2} \leqq \frac{\pi}{2}(n+1) + \frac{\pi^2}{2\epsilon\alpha}.$$

Thus we have

$$\frac{1}{n}\int_0^\epsilon \psi_1 dt \leqq \frac{\pi^2}{8} + \frac{\pi}{2} + \frac{\pi^2}{2n\epsilon\alpha},$$

from which we see that the first term on the right in (12) has the property indicated of (8). Likewise, the same is seen to be true of the second term on the right in (12) with which the proof becomes complete.

The proof of (II)′ of § 47 follows by considering the special case in which $\alpha = 1$, $\beta = \frac{1}{2}$.

4. *Lemma V. With k defined as in Lemma IV we have*

(15) $$\lim_{n=\infty}\frac{1}{n}\int_{-\epsilon}^\epsilon\left|\sum_{n=0}^{n}\cos kt\right|dt = 0,$$

where ϵ is any positive constant such that $\epsilon < 1$, $\epsilon < \pi/\alpha$.

As in the study of (8) it will here suffice to prove the lemma for the expression obtained from (15) by replacing the $-\epsilon$ of the lower limit of integration by 0.

Now, we have

$$\cos kt = \cos n\alpha t\cos\beta t - \sin n\alpha t\sin\beta t,$$

so that upon using formulae (10) and (11) we obtain

$$\frac{1}{n}\int_0^\epsilon\left|\sum_{n=0}^{n}\cos kt\right|dt < \frac{2}{n}\int_0^\epsilon H dt,$$

where

$$H = \frac{\sin\frac{(n+1)\alpha t}{2}}{\ldots}.$$

For the given value of ϵ we may take n so large that

$$\frac{\pi}{(n+1)\alpha} < \epsilon$$

and write

(16)
$$\int_0^\epsilon H dt = \int_0^{\pi/(n+1)\alpha} H dt + \int_{\pi/(n+1)\alpha}^\epsilon H dt.$$

From (14) it appears that the first term here appearing on the right is less than
Again, the second term on the right in (16) may be put into the form

$$\frac{2}{\alpha}\int_{\pi/(n+1)\alpha}^\epsilon \sin\frac{n\alpha t}{2}\left(\frac{\frac{\alpha t}{2}}{\sin\frac{\alpha t}{2}}\right)\frac{dt}{t},$$

which is less than

$$\frac{2}{\alpha}\int_{\pi/(n+1)\alpha}^\epsilon \left(\frac{\pi}{2}\right)\frac{dt}{t} = \frac{\pi}{\alpha}\log\frac{(n+1)\epsilon\alpha}{\pi}.$$

Thus we have

$$\frac{2}{n}\int_0^\epsilon H dt] < \frac{\pi^2}{n\alpha} + \frac{2}{n\alpha}\log\frac{(n+1)\epsilon\alpha}{\pi},$$

from which the truth of the lemma becomes evident.

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